Седьмая всероссийская конференция
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Монотонные разностные схемы сквозного счёта, сохраняющие повышенную точность в областях влияния ударных волн

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Hyperbolic Systems of Conservation Laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0 \qquad (1)$$

- Aerospace branch
- Nuclear branch
- Hydrodynamics of opened channel
- Film flows
- Problems of elasticity and plasticity

Monotone Finite-Difference Schemes

Exact solution of Cauchy problem for the linear transport equation

$$u_t + u_x = 0, \quad u(x,0) = v(x) \quad \Rightarrow \quad u(x,0) = v(x-t) \tag{2}$$

Definition 1 (*Godunov*)

Explicit two-layer in time finite-difference scheme

$$u_j^{n+1} = \sum_k c_k u_{j+k}^n \tag{3}$$

approximating equation (2) *is monotone if it turns every monotonic on j function u_j^n to monotonic on j function u_j^{n+1} with the same sign of monotonicity.*

Theorem 1 (*Godunov*) Explicit two-layer in time finite-difference scheme is monotone if and only if

$$c_k \geq 0 \quad \forall k \tag{4}$$

Godunov S.K. A difference Method for Numerical Calculation of Discontinuous Solutions of the Equations of Hydrodynamics, *Mat. Sb.* 1959

Godunov's taboo

There are no monotone finite-difference schemes
(with smooth numerical flux functions)
higher than the first order of approximation

Godunov S.K. A difference Method for Numerical Calculation of Discontinuous Solutions of the Equations of Hydrodynamics, *Mat. Sb.* 1959.

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Attempts to overcome Godunov's taboo

NFC-like schemes (Nonlinear Flux Correction)

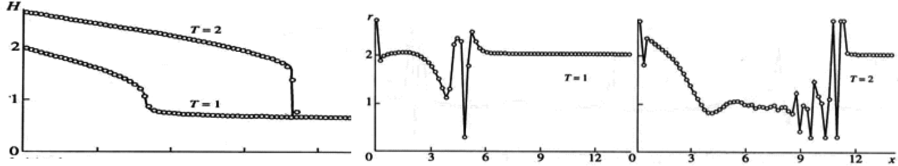
- *Goldin, Kalitkin, Shitova*, 1965.
- FCT, *Boris, Book*, 1975.
- *Kolgan*, 1978.
- MUSCL, *Van Leer*, 1979.
- TVD schemes, *Harten*, 1983.
- ENO schemes, *Harten, Osher*, 1987.
- NED schemes, *Tadmor*, 1990.
- WENO schemes, *Liu, Osher, Chan*, 1994; *Jiang, Shu*, 1996.
- CABARET schemes, *Samarskii, Goloviznin, Karabasov*, 1998, 2005.

Real Accuracy of NFC-schemes

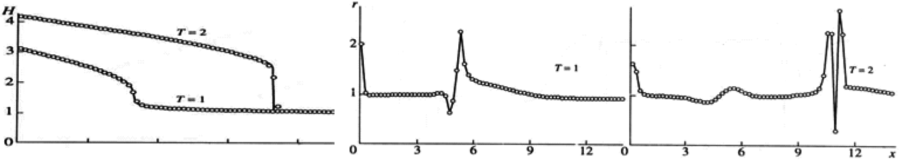
1. **Ostapenko V.V.** On convergence of difference schemes behind of nonstationary shock, *Comp. Maths Math. Phys.* 1997.
2. **Casper J., Carpenter M.N.** Computational consideration for the simulation of shock-induced sound, *SIAM J. Sci. Comput.* 1998.
3. **Engquist B., Sjogreen B.** The convergence rate of finite difference schemes in the presence of shocks, *SIAM J. Numer. Anal.* 1998.
4. **Kovyrkina O.A., Ostapenko V.V.** On the convergence of shock-capturing difference schemes, *Dokl. Math.* 2010.
5. **Kovyrkina O.A., Ostapenko V.V.** On the practical accuracy of shock-capturing schemes, *Math. Models. Comput. Simul.* 2014.
6. **Kovyrkina O.A., Kudryavtsev A.N., Ostapenko V.V.** On real accuracy of WENO schemes at shock capturing calculations, *International conference «AMCA – 2014», Novosibirsk, Russia.*
7. **Mikhailov N.A.** The convergence order of WENO schemes behind a shock front, *Math. Models. Comput. Simul.* 2015.

Finite-Difference Schemes of "High Accuracy"

TVD scheme of formal second order (*Harten*, 1983)



First order scheme with artificial viscosity (*Ostapenko*, 1987)



1. **Ostapenko V.V.** On convergence of difference schemes behind of nonstationary shock, *Comp. Maths Math. Phys.* 1997.
2. **Casper J., Carpenter M.N.** Computational consideration for the simulation of shock-induced sound, *SIAM J. Sci. Comput.* 1998.
3. **Engquist B., Sjogreen B.** The convergence rate of finite difference schemes in the presence of shocks, *SIAM J. Numer. Anal.* 1998.

Runge Method for Experimental Determination of Schemes Convergence

$$\boldsymbol{v}_i(x,t) - \boldsymbol{u}(x,t) = \boldsymbol{C}\Delta_i^k + o(\Delta_i^k) \tag{5}$$

$\boldsymbol{u}$ — exact solution,
$\boldsymbol{v}_i$ — numerical solution,
k — order of convergence.

Calculations are held on the sequence of three embedded grids with the space steps

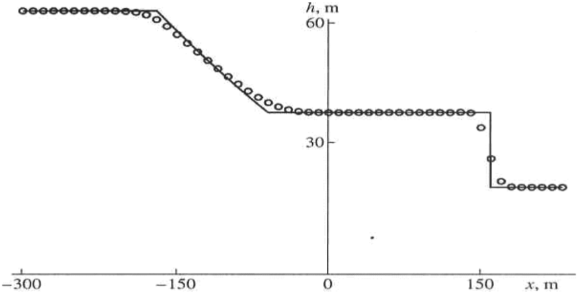
$$\Delta_0 = \Delta, \quad \Delta_1 = \Delta/2, \quad \Delta_2 = \Delta/4$$

$$\delta\boldsymbol{v}_{i,i+1} = \boldsymbol{v}_i - \boldsymbol{v}_{i+1}, \quad i = 0,1 \tag{6}$$

$$\frac{|\delta\boldsymbol{v}_{0,1}|}{|\delta\boldsymbol{v}_{1,2}|} = \frac{\Delta_1^k - \Delta_2^k}{\Delta_0^k - \Delta_1^k} = \left(\frac{1}{2}\right)^k \quad \Rightarrow \quad k = \log_{1/2}\frac{|\delta\boldsymbol{v}_{1,2}|}{|\delta\boldsymbol{v}_{0,1}|} \tag{7}$$

Why so long time was it widely spread a misconception that NFC schemes conserve high order of convergence in all smooth parts of calculated weak solutions?

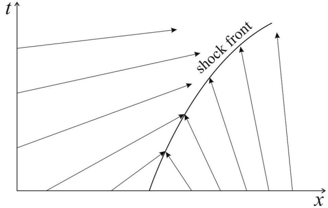
1. In majority of papers due to the construction of high order shock capturing schemes the scheme accuracy have tested as resolution of Riemann problem in which arise only stationary shocks with a constant solution behind them.

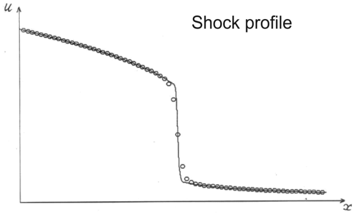


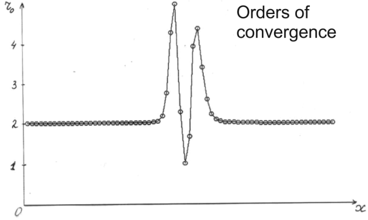
Dam break problem

Why so long time was it widely spread a misconception that NFC schemes conserve high order of convergence in all smooth parts of calculated weak solutions?

2. For scalar conservation law it is absent transmitted characteristics field and the domain of shock influence coincide with the shock front.







Nonstationary shock for nonlinear transport equation $u_t + (u^2/2)_x = 0$

ε-Rankine-Hugoniot Conditions at the Shocks

$$[D\boldsymbol{u} - \boldsymbol{f}(\boldsymbol{u})]_\varepsilon + \frac{d}{dt} \int\limits_{x(t)-\varepsilon}^{x(t)+\varepsilon} \boldsymbol{u}\, dx = 0 \tag{8}$$

$$[\boldsymbol{f}(t,x)]_\varepsilon = \boldsymbol{f}(t, x(t) - \varepsilon) - \boldsymbol{f}(t, x(t) + \varepsilon) \tag{9}$$

Theorem 2.

If difference scheme has smooth difference fluxes then its approximation order of ε-Rankine-Hugoniot conditions agree with its classical approximation order on smooth solutions.

In NFC schemes difference fluxes are defined by virtue of different minimax procedures and as a result all this fluxes are not smooth enough. So NFC type schemes approximate ε-Rankine-Hugoniot conditions (8) with the order not higher then the first.

Ostapenko V.V. On finite-difference approximation of Hugoniot conditions on shock whish propagate with variable velocity, Comput. Math. Math. Phys. 1998.

Definition of Integral Convergence

Let's set the number $a \in \mathbb{R}$ and define the integrals

$$\boldsymbol{U}^a(t,x) = \int_x^b \boldsymbol{u}(t,y)dy, \quad \boldsymbol{V}_i^a(t,x) = \int_x^a \boldsymbol{v}_i(t,y)dy$$

Definition 2.

The sequence of difference solutions $\boldsymbol{v}_i(t,x)$ converges on the interval $[x,a] \subset \mathbb{R}$ with the Rth order $(0 < R \leq 2)$, to the exact solution $\boldsymbol{u}(t,x)$, if

$$\boldsymbol{V}_i^a(t,x) - \boldsymbol{U}^a(t,x) = \boldsymbol{C}\Delta_i^R + o\left(\Delta_i^R\right),$$

where the vector function $\boldsymbol{C}$ is independent of Δ_i.

$$\Delta_0 = \Delta, \quad \Delta_1 = \Delta/2, \quad \Delta_2 = \Delta/4$$

$$\delta\boldsymbol{V}_{i,i+1} = \boldsymbol{V}_i^a - \boldsymbol{V}_{i+1}^a = \boldsymbol{C}(\Delta_i^R - \Delta_{i+1}^R), \quad i = 0, 1.$$

$$\frac{|\delta\boldsymbol{V}_{1,2}|}{|\delta\boldsymbol{V}_{0,1}|} = \frac{\Delta_1^R - \Delta_2^R}{\Delta_0^R - \Delta_1^R} = \left(\frac{1}{2}\right)^R \quad \Rightarrow \quad R = \log_{1/2}\frac{|\delta\boldsymbol{V}_{1,2}|}{|\delta\boldsymbol{V}_{0,1}|}$$

Kovyrkina O.A., Ostapenko V.V. On the convergence of shock-capturing difference schemes, *Dokl. Math.* 2010.

Periodic Initial Value Problem for Shallow Water Equations

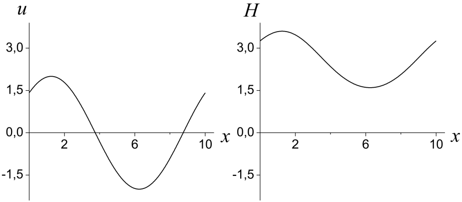
$$\begin{cases} h_t + q_x = 0 \\ q_t + (qu + gh^2/2)_x = 0 \end{cases} \tag{10}$$

$$u(x,0) = \beta \sin\left(\frac{2\pi x}{X} + \frac{\pi}{4}\right) \tag{11}$$

$$h(x,0) = \frac{(u(x,0)+\theta)^2}{4g} =$$

$$= \frac{1}{4g}\left(\beta \sin\left(\frac{2\pi x}{X} + \frac{\pi}{4}\right) + \theta\right)^2 \tag{12}$$

$$X = \theta = 10,\ \ \beta = 2.$$



initial values

The initial values of the invariants

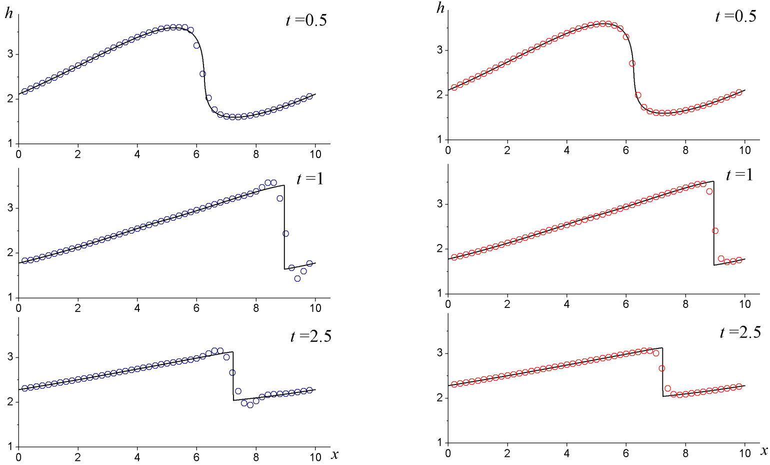
$$w_1(x,0) = -\theta = \text{const}, \quad w_2(x,0) = 2u(x,0) + \theta \tag{13}$$

$$w_1 = u - 2c, \quad w_2 = u + 2c, \quad c = \sqrt{gh} \tag{14}$$

Kovyrkina O. A., Ostapenko V. V. On the practical accuracy of shock-capturing schemes, *Math. Model. Comput. Simul.* 2014.

Comparison of "Exact" and Numerical Solutions

calculations on the spatial interval $[iX, (i+1)X]$ at $i = 0$, $\Delta = 0.2$

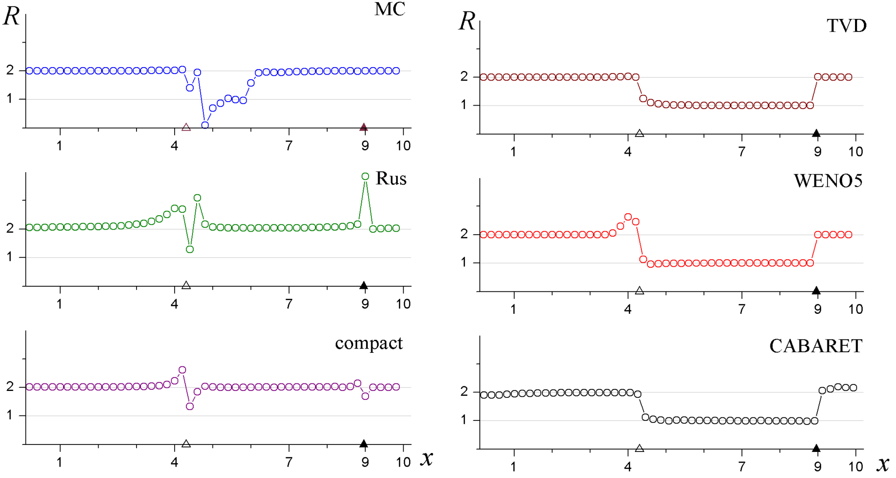


Rusanov V.V. Difference schemes of third order accuracy for the through calculation of discontinuous solutions, *Dokl. Akad. Nauk SSSR.* 1968.

Jiang G.S., Shu C.W. Efficient implementation of weighted ENO schemes, *J. Comput. Phys.* 1996.

Integral Orders of Convergence, $t = 1$

Nonmonotonic schemes | NFC schemes



space step $\Delta = 0.004$, every 50th grid node is shown

Kovyrkina O.A., Kudryavtsev A.N., Ostapenko V.V. On real accuracy of WENO schemes at shock capturing calculations, *International conference «AMCA – 2014», Russia.*

Why are Nonmonotonic Schemes with Higher Order Transfer the Rankine-Hugoniot Conditions through the Shock?

The **oscillations** arising on shock wave fronts in classical nonmonotonic schemes of high accuracy **keep information** about Fourier wave structure of the expansion of a discontinuous function in the vicinity of a strong discontinuity, which allows to these schemes with high accuracy transfer Rankine-Hugoniot conditions through the smeared shock wave fronts and maintain increased accuracy in the regions of shock wave influence.
At the same time NFC schemes, by smoothing these oscillations, lose this information, which leads to a decrease in their accuracy in the transfer of Rankine-Hugoniot conditions.

Alternative in the Theory of Finite-Difference Schemes

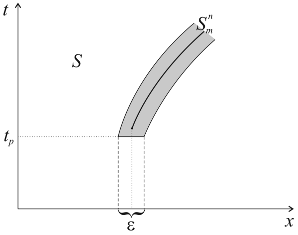
Is it not possible

to localize a shock wave front with higher accuracy
and, at the same time,
maintain an increased order of convergence
in the domain of influence
of the shock wave?

Combined Finite-Difference Schemes

$$S = \{(x_j, t_n): \ x_j = j\Delta, \ 0 \le j \le N; \quad t_{n+1} = t_n + \tau_n, \ t_0 = 0\}$$

$$S_m^n = \left\{(x_j, t_n) : j_n - m \le j \le j_n + m + 1, \ \left|\mathbf{u}_{j_n+1}^n - \mathbf{u}_{j_n}^n\right| = \max_j \left|\mathbf{u}_{j+1}^n - \mathbf{u}_j^n\right| \ge p\Delta\right\}$$



The number p determines the beginning of the formation of a numerical shock wave.
To eliminate oscillations in the region S_m^n we reconstruct the difference solution $\mathbf{u}_j^n$ by replacing it to the monotonic solution $\mathbf{v}_j^n$ obtained as a result of numerical calculation in this region by the internal scheme of the initial-boundary value problem for system (1). The initial and boundary conditions for this inner problem are taken from the difference solution $\mathbf{u}_j^n$ obtained by a basic scheme.

$$\mathbf{w}_j^n = \begin{cases} \mathbf{u}_j^n, & (x_j, t_n) \in S \backslash S_m^n \\ \mathbf{v}_j^n, & (x_j, t_n) \in S_m^n \end{cases} \qquad p = 1, \ m = 6$$

Examples of Combined Schemes

Basic nonmonotonic 3th order scheme:
- ✓ explicit Rusanov scheme [1];
- ✓ implicit compact scheme [2]

Internal monotonic 2th order scheme: modification of the CABARET scheme [3]

$$\begin{cases} h_t + q_x = 0 \\ q_t + (qu + gh^2/2)_x = 0 \end{cases}$$

Initial-boundary value problem

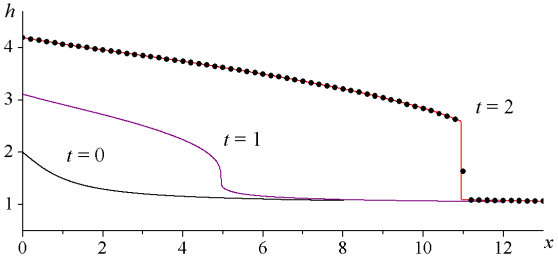
$$h(0,x) = 2 - \frac{2}{\pi}\operatorname{arctg} x$$
$$q(0,x) = 0,\ q(t,0) = \alpha t$$
$$\alpha = 4g/\pi$$

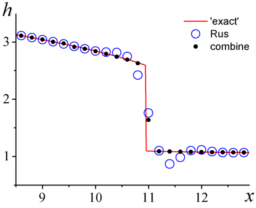
Cauchy problem (10)–(12)

$$u(x,0) = \beta \sin\left(\frac{2\pi x}{X} + \frac{\pi}{4}\right)$$
$$h(x,0) = \left(u(x,0) + \theta\right)^2 / (4g)$$
$$X = \theta = 10,\ \beta = 2.$$

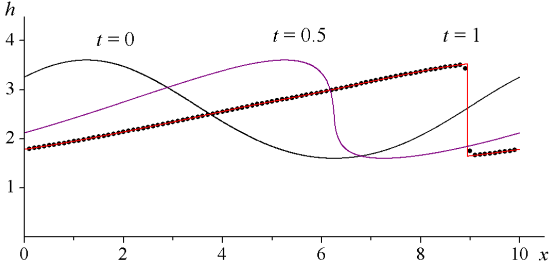
1. **Rusanov V.V.** Difference schemes of third order accuracy for the through calculation of discontinuous solutions, *Dokl. Akad. Nauk SSSR.* 1968.
2. **Ostapenko V.V.** Construction of high-order accurate shock-capturing finite-difference schemes for unsteady shock waves, *Comput. Maths. Math. Phys.* 2000.
3. **Karabasov S.A., Goloviznin V.M.** Compact Accurately Boundary Adjusting high-REsolution Technique for Fluid Dynamics, *J. Comput. Phys.* 2009
4. **Kovyrkina O.A., Ostapenko V.V.** On the construction of combined finite-difference schemes of high accuracy, *Dokl. Math.* 2018.
5. **Zyuzina N.A., Kovyrkina O.A., Ostapenko V.V.** Monotone finite-difference scheme that preserves the high accuracy in the regions of shock influence, *Dokl. Math.* (in print)

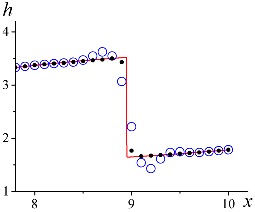
Calculations by combined scheme





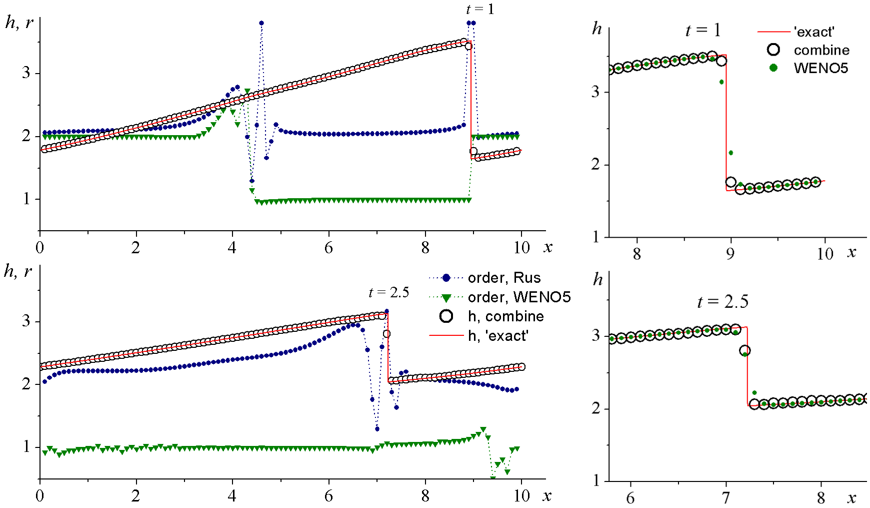
initial-boundary value problem





Cauchy problem

Integral Orders for Cauchy Problem



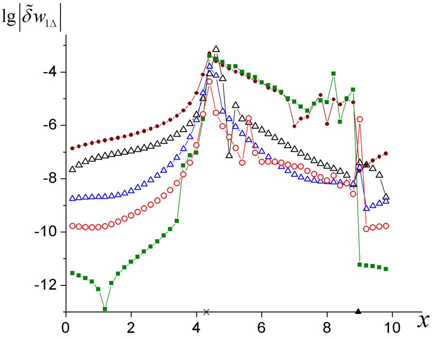
h: $\Delta = 0.1$, every grid node is shown

orders: $\Delta = 0.005$, every 20th grid node is shown

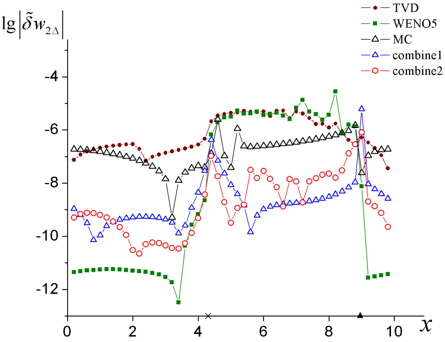
Relative Errors of Evaluation of Riemann Invariants at $t = 1$

$$\bar{\delta}\varphi_{k,k+1} = \varphi_k - \varphi_{k+1}, \quad \delta\varphi = \frac{\bar{\delta}\varphi_{0,1}}{1 - |\bar{\delta}\varphi_{1,2}|/|\bar{\delta}\varphi_{0,1}|} \quad \Rightarrow \quad \tilde{\delta}\varphi_{\Delta} = \frac{\delta\varphi}{\varphi}$$

$w_1 = u - 2\sqrt{gh}$



$w_2 = u + 2\sqrt{gh}$



space step $\Delta = 0.004$, every 50th grid node is shown

Conclusion

- The method is proposed for constructing combined shock-capturing finite-difference schemes, which with high accuracy capture the shocks and simultaneously maintain an increased convergence order in all domains of smoothness of the calculated weak solutions.
- The concrete combined schemes are considered, where nonmonotonic third-order scheme (Rusanov or compact) is used as the basic scheme, and as the inner one is a monotone CABARET scheme of the second order of accuracy for smooth solutions.
- We presented the test calculations that demonstrate the advantages of the new schemes.

СПАСИБО

ЗА ВНИМАНИЕ!