A new staggered semi-implicit discontinuous Galerkin scheme for the shallow water and the incompressible Navier-Stokes equations

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ERC Grant *STiMulUs*

European Research Council

Svetlogorsk, Russia, 25/09/2016

Standard Approaches to Hydrostatic Free Surface Flows

1. Godunov-type finite volume schemes for the single or multi-layer shallow water equations (depth-integrated incompressible Navier-Stokes equations using the assumption of a **hydrostatic** pressure distribution). Most frequently used in the community on hyperbolic conservation laws

\+ Easy implementation on **collocated** structured and unstructured grids
\+ High order of accuracy together with discrete maximum principle or at least essentially non-oscillatory behaviour can be easily achieved in space and time using high resolution TVD schemes or ENO/WENO methods with TVD-RK time stepping
\+ Exact conservation of mass and momentum and any other conserved quantity

\- CFL type time step restriction, resulting in very small time steps for typical geophysical problems
\- Computationally expensive
\- Other practical problems: wetting and drying and well-balancing

Standard Approaches to Hydrostatic Free Surface Flows

2. Semi-implicit finite volume & finite difference schemes for the shallow water equations **(UnTRIM²)** of Casulli et al.

\+ Easy implementation on **staggered** orthogonal unstructured grids
\+ Exact conservation of mass
\+ Exactly well-balanced
\+ Very large time steps, **unbeatable** computational efficiency for geophysical flows
\+ **Rigorous** treatment of wetting and drying (**no parameter**, **proven convergence**!)
\+ Available extensions to 3D hydrostatic and to fully 3D non-hydrostatic flows

\- Low order accurate (≤ 2)

Aim of the present work: Extension of TRIM to arbitrary high order of accuracy *N* in space (TRIM^N), reducing to TRIM in the special case of *N=0*.

A New High Order Semi-Implicit Staggered DG FEM method

Governing equations written in conservative variables

$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + gH\frac{\partial \eta}{\partial x} = 0,$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} = 0,$$

$\eta = \eta(x,t)$: free surface
$u = u(x,t)$: flow velocity
h=h(x): known bottom bathymetry
H=H(x,t): water depth
U=Hu: flow rate

A New High Order Semi-Implicit Staggered DG FEM method

The discrete numerical solution is represented in terms of piecewise polynomials of degree N that are allowed to **jump** across element boundaries (**not** in standard FEM):

$$U_{i+\frac{1}{2}}(x,t) = \boldsymbol{\phi}(x) \cdot \hat{\mathbf{U}}_{i+\frac{1}{2}}(t), \quad u_{i+\frac{1}{2}}(x,t) = \boldsymbol{\phi}(x) \cdot \hat{\mathbf{u}}_{i+\frac{1}{2}}(t),$$

$$H_{i+\frac{1}{2}}(x,t) = \boldsymbol{\phi}(x) \cdot \hat{\mathbf{H}}_{i+\frac{1}{2}}(t), \quad \eta_i(x,t) = \boldsymbol{\omega}(x) \cdot \hat{\boldsymbol{\eta}}_i(t),$$

with the (known) sets of spatial basis functions

$$\boldsymbol{\phi}(x) = (\phi_0(x), \phi_1(x), \ldots, \phi_N(x))$$

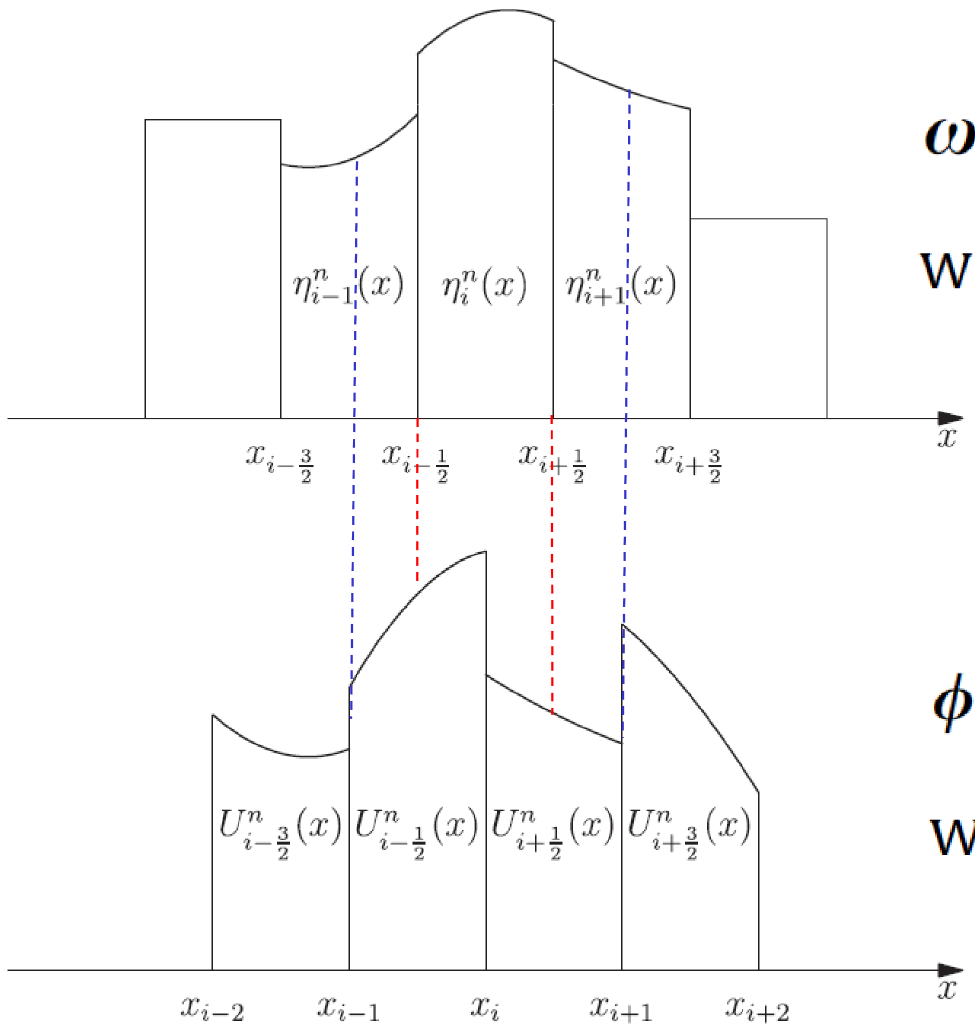
$$\boldsymbol{\omega}(x) = (\omega_0(x), \omega_1(x), \ldots, \omega_N(x))$$

and the vectors of the (unknown) time-dependent degrees of freedom (coefficients)

$$\hat{\mathbf{U}}_{i+\frac{1}{2}}(t) \quad \hat{\mathbf{u}}_{i+\frac{1}{2}}(t) \quad \hat{\mathbf{H}}_{i+\frac{1}{2}}(t) \quad \hat{\boldsymbol{\eta}}_i(t)$$

A New High Order Semi-Implicit Staggered DG FEM method

Use of a **staggered** grid for velocity and pressure (free surface):



$$\omega(x) = \varphi(\xi),$$

$$\text{with } x = x_{i-\frac{1}{2}} + \xi\,\Delta x, \quad 0 \leqslant \xi \leqslant 1$$

$$\phi(x) = \varphi(\xi),$$

$$\text{with } x = x_i + \xi\,\Delta x, \quad 0 \leqslant \xi \leqslant 1$$

A New High Order Semi-Implicit Staggered DG FEM method

Multiplication of the governing PDE system with test functions and integration over the respective control volumes yields:

$$\int_{x_i}^{x_{i+1}} \phi \left(\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + gH \frac{\partial \eta}{\partial x} \right) dx = 0,$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \omega \left(\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} \right) dx = 0.$$

In the first integral (momentum equation), we need to take into account properly the discontinuity of the free surface elevation η at $x_{i+1/2}$. The second term in the second integral (mass flux in the mass conservation equation) is written in flux divergence form and can be integrated by parts.

Piecewise high order data representation can be also interpreted as a high order subgrid method.

A New High Order Semi-Implicit Staggered DG FEM method

This yields the discrete momentum equation

$$\int_{x_i}^{x_{i+1}} \phi \left(\frac{\partial U}{\partial t} + \frac{\partial u U}{\partial x} \right) dx + \phi_{i+\frac{1}{2}} g H(x_{i+\frac{1}{2}}, t)(\eta^+_{i+\frac{1}{2}} - \eta^-_{i+\frac{1}{2}}) + \int_{x_i}^{x^-_{i+\frac{1}{2}}} \phi g H \frac{\partial \eta}{\partial x} dx + \int_{x^+_{i+\frac{1}{2}}}^{x_{i+1}} \phi g H \frac{\partial \eta}{\partial x} dx = 0,$$

and the discrete mass conservation equation

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \omega \frac{\partial \eta}{\partial t} dx + \omega_{i+\frac{1}{2}} U(x_{i+\frac{1}{2}}, t) - \omega_{i-\frac{1}{2}} U(x_{i-\frac{1}{2}}, t) - \int_{x^+_{i-\frac{1}{2}}}^{x^-_{i+\frac{1}{2}}} \frac{\partial \omega}{\partial x} U \, dx = 0.$$

Inserting the discrete solution and using the change of coordinates to the unit reference element [0;1] yields the following scheme, where the nonlinear convective terms are discretized explicitly, the total water depth is taken at time n and all the other terms are taken at time $t = t^n + \theta \Delta t$.

The method is locally and globally mass conservative (proof by setting $\phi = 1$).

A New High Order Semi-Implicit Staggered DG FEM method

$$\frac{\Delta x}{\Delta t}\left[\int_0^1 \varphi(\xi)\varphi(\xi)d\xi\right]\cdot\left(\hat{\mathbf{U}}^{n+1}_{i+\frac{1}{2}}-\widehat{\mathbf{FU}}^{n}_{i+\frac{1}{2}}\right)+g\varphi\left(\frac{1}{2}\right)\varphi\left(\frac{1}{2}\right)\cdot\hat{\mathbf{H}}^{n}_{i+\frac{1}{2}}\left[\varphi(0)\cdot\hat{\boldsymbol{\eta}}^{n+\theta}_{i+1}-\varphi(1)\cdot\hat{\boldsymbol{\eta}}^{n+\theta}_{i}\right]$$
$$+g\left[\int_0^{\frac{1}{2}} \varphi(\xi)\varphi(\xi)\varphi'\left(\xi+\frac{1}{2}\right)d\xi\right]\cdot\hat{\mathbf{H}}^{n}_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}^{n+\theta}_{i}+g\left[\int_{\frac{1}{2}}^{1} \varphi(\xi)\varphi(\xi)\varphi'\left(\xi-\frac{1}{2}\right)d\xi\right]\cdot\hat{\mathbf{H}}^{n}_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}^{n+\theta}_{i+1}=0,$$

$$\frac{\Delta x}{\Delta t}\left[\int_0^1 \varphi(\xi)\varphi(\xi)d\xi\right]\cdot\left(\hat{\boldsymbol{\eta}}^{n+1}_{i}-\hat{\boldsymbol{\eta}}^{n}_{i}\right)+\varphi(1)\varphi\left(\frac{1}{2}\right)\cdot\hat{\mathbf{U}}^{n+\theta}_{i+\frac{1}{2}}-\varphi(0)\varphi\left(\frac{1}{2}\right)\cdot\hat{\mathbf{U}}^{n+\theta}_{i-\frac{1}{2}}-\left[\int_0^{\frac{1}{2}} \varphi'(\xi)\varphi\left(\xi+\frac{1}{2}\right)d\xi\right]\cdot\hat{\mathbf{U}}^{n+\theta}_{i-\frac{1}{2}}$$
$$-\left[\int_{\frac{1}{2}}^{1} \varphi'(\xi)\varphi\left(\xi-\frac{1}{2}\right)d\xi\right]\cdot\hat{\mathbf{U}}^{n+\theta}_{i+\frac{1}{2}}=0,$$

Due to the use of a staggered grid, in principle all the terms above are well defined and no Riemann solver has to be employed at the element interfaces. However, in the presence of strong discontinuities either the use of a Riemann solver (Rusanov-type flux) and /or the use of a limiter is highly recommended to reduce spurious oscillations created at the shock wave.

A New High Order Semi-Implicit Staggered DG FEM method

The scheme can be written in a more compact matrix-vector form, which looks formally very similar and for *N=0* is **identical** to the original semi-implicit method of [Casulli 1990]

$$\mathbf{M} \cdot \left(\hat{\mathbf{U}}_{i+\frac{1}{2}}^{n+1} - \widehat{\mathbf{FU}}_{i+\frac{1}{2}}^{n} \right) + g \frac{\Delta t}{\Delta x} \left(\mathbf{R}_u \cdot \hat{\mathbf{H}}_{i+\frac{1}{2}}^{n} \hat{\boldsymbol{\eta}}_{i+1}^{n+\theta} - \mathbf{L}_u \cdot \hat{\mathbf{H}}_{i+\frac{1}{2}}^{n} \hat{\boldsymbol{\eta}}_{i}^{n+\theta} \right) = 0,$$

$$\mathbf{M} \cdot \left(\hat{\boldsymbol{\eta}}_{i}^{n+1} - \hat{\boldsymbol{\eta}}_{i}^{n} \right) + \frac{\Delta t}{\Delta x} \left(\mathbf{R}_\eta \cdot \hat{\mathbf{U}}_{i+\frac{1}{2}}^{n+\theta} - \mathbf{L}_\eta \cdot \hat{\mathbf{U}}_{i-\frac{1}{2}}^{n+\theta} \right) = 0.$$

after introducing the following integrals, which can all be precomputed

$$\mathbf{R}_u = \boldsymbol{\varphi}\left(\frac{1}{2}\right)\boldsymbol{\varphi}\left(\frac{1}{2}\right)\boldsymbol{\varphi}(0) + \int_{\frac{1}{2}}^{1} \boldsymbol{\varphi}(\xi)\boldsymbol{\varphi}(\xi)\boldsymbol{\varphi}'(\xi - \frac{1}{2})d\xi, \qquad \mathbf{L}_u = \boldsymbol{\varphi}\left(\frac{1}{2}\right)\boldsymbol{\varphi}\left(\frac{1}{2}\right)\boldsymbol{\varphi}(1) - \int_{0}^{\frac{1}{2}} \boldsymbol{\varphi}(\xi)\boldsymbol{\varphi}(\xi)\boldsymbol{\varphi}'\left(\xi + \frac{1}{2}\right)d\xi,$$

$$\mathbf{R}_\eta = \boldsymbol{\varphi}(1)\boldsymbol{\varphi}\left(\frac{1}{2}\right) - \int_{\frac{1}{2}}^{1} \boldsymbol{\varphi}'(\xi)\boldsymbol{\varphi}\left(\xi - \frac{1}{2}\right)d\xi \qquad \mathbf{L}_\eta = \boldsymbol{\varphi}(0)\boldsymbol{\varphi}\left(\frac{1}{2}\right) + \int_{0}^{\frac{1}{2}} \boldsymbol{\varphi}'(\xi)\boldsymbol{\varphi}\left(\xi + \frac{1}{2}\right)d\xi,$$

$$\mathbf{M} = \int_{0}^{1} \boldsymbol{\varphi}(\xi)\boldsymbol{\varphi}(\xi)\, d\xi,$$

A New High Order Semi-Implicit Staggered DG FEM method

As in the original TRIM method, the momentum equation is substituted into the continuity equation to yield the following discrete wave equation for the free surface η:

$$\mathbf{M}\cdot\hat{\boldsymbol{\eta}}_i^{n+1} - g\theta^2\frac{\Delta t^2}{\Delta x^2}\left[\mathbf{R}_\eta\cdot\mathbf{M}^{-1}\cdot\left(\mathbf{R}_u\cdot\hat{\mathbf{H}}^n_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i+1}^{n+1} - \mathbf{L}_u\cdot\hat{\mathbf{H}}^n_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i}^{n+1}\right) - \mathbf{L}_\eta\cdot\mathbf{M}^{-1}\cdot\left(\mathbf{R}_u\cdot\hat{\mathbf{H}}^n_{i-\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i}^{n+1} - \mathbf{L}_u\cdot\hat{\mathbf{H}}^n_{i-\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i-1}^{n+1}\right)\right] = \mathbf{b}_i^n$$

$$\mathbf{b}_i^n = \mathbf{M}\cdot\hat{\boldsymbol{\eta}}_i^{n} - \frac{\Delta t}{\Delta x}\left[\mathbf{R}_\eta\cdot\widehat{\mathbf{FU}}^{n+\theta}_{i+\frac{1}{2}} - \mathbf{L}_\eta\cdot\widehat{\mathbf{FU}}^{n+\theta}_{i-\frac{1}{2}}\right] + g\theta(1-\theta)\frac{\Delta t^2}{\Delta x^2}\left[\mathbf{R}_\eta\cdot\mathbf{M}^{-1}\cdot\left(\mathbf{R}_u\cdot\hat{\mathbf{H}}^n_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i+1}^{n} - \mathbf{L}_u\cdot\hat{\mathbf{H}}^n_{i+\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i}^{n}\right) - \mathbf{L}_\eta\cdot\mathbf{M}^{-1}\cdot\left(\mathbf{R}_u\cdot\hat{\mathbf{H}}^n_{i-\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i}^{n} - \mathbf{L}_u\cdot\hat{\mathbf{H}}^n_{i-\frac{1}{2}}\hat{\boldsymbol{\eta}}_{i-1}^{n}\right)\right]$$

Small stencil (S_i = {i-1,i,i+1}). The linear algebraic system is solved using a preconditioned GMRES method [Saad & Schultz, 1986].

A New High Order Semi-Implicit Staggered DG FEM method

Computation of ***H*** via L_2 projection:

$$\hat{\mathbf{H}}^n_{i+\frac{1}{2}} = \hat{\mathbf{h}}_{i+\frac{1}{2}} + \mathbf{M}^{-1} \cdot \left(\mathbf{M}_L \cdot \hat{\boldsymbol{\eta}}^n_i + \mathbf{M}_R \cdot \hat{\boldsymbol{\eta}}^n_{i+1} \right)$$

$$\mathbf{M}_L = \int_0^{\frac{1}{2}} \boldsymbol{\varphi}(\xi) \boldsymbol{\varphi}(\xi + \frac{1}{2})\, d\xi, \quad \mathbf{M}_R = \int_{\frac{1}{2}}^{1} \boldsymbol{\varphi}(\xi) \boldsymbol{\varphi}\left(\xi - \frac{1}{2}\right) d\xi.$$

The nonlinear convective terms ***Fu*** are computed with a classical collocated grid DG FEM approach on the pressure grid using a Rusanov-type upwind flux:

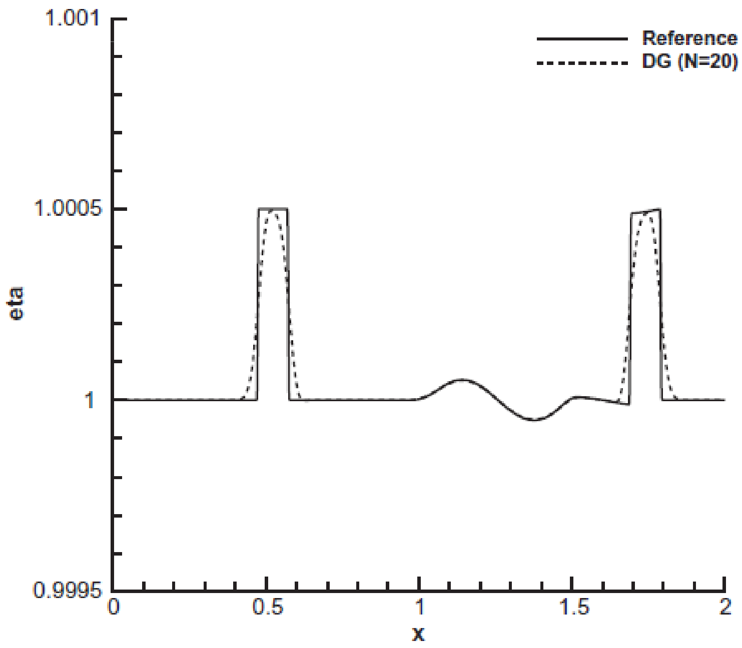
$$\widehat{\mathbf{FU}}^n_{i+\frac{1}{2}} = \mathbf{M}^{-1} \cdot \left(\mathbf{M}_L \cdot \widehat{\mathbf{FU}}^n_i + \mathbf{M}_R \cdot \widehat{\mathbf{FU}}^n_{i+1} \right) \qquad \hat{\mathbf{U}}^n_i = \mathbf{M}^{-1} \cdot \left(\mathbf{M}_L \cdot \hat{\mathbf{U}}^n_{i-\frac{1}{2}} + \mathbf{M}_R \cdot \hat{\mathbf{U}}^n_{i+\frac{1}{2}} \right),$$

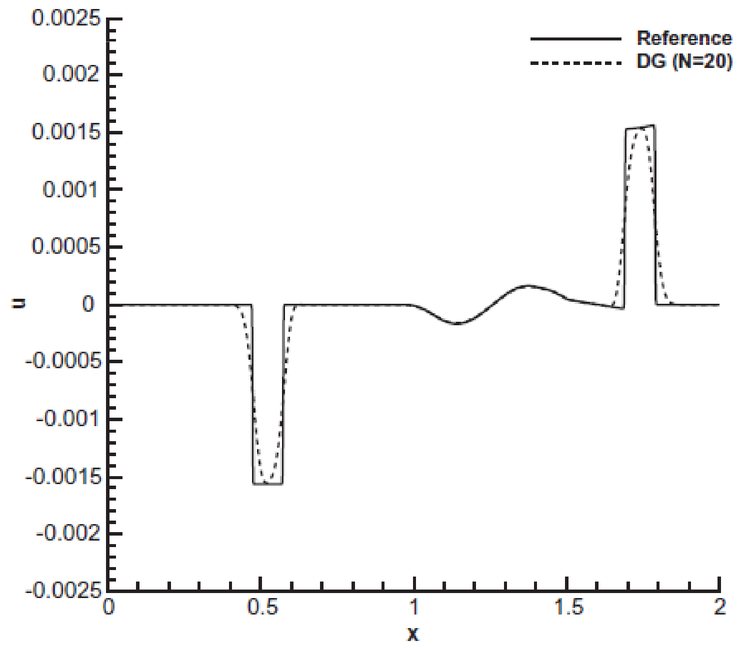
$$\widehat{\mathbf{FU}}^n_i = \hat{\mathbf{U}}^n_i - \frac{\Delta t}{\Delta x} \mathbf{M}^{-1} \cdot \left[\boldsymbol{\varphi}(1) f^n_{i+\frac{1}{2}} - \boldsymbol{\varphi}(0) f^n_{i-\frac{1}{2}} - \int_0^1 \boldsymbol{\varphi}'(\xi) \boldsymbol{\varphi}(\xi)\, d\xi \cdot \hat{\mathbf{f}}^n_i \right]$$

$$f^n_{i+\frac{1}{2}} = \frac{1}{2} \left[\boldsymbol{\varphi}(0) \cdot \hat{\mathbf{f}}^n_{i+1} + \boldsymbol{\varphi}(1) \cdot \hat{\mathbf{f}}^n_i \right] - \frac{1}{2} s^n_{i+\frac{1}{2}} \left[\boldsymbol{\varphi}(0) \cdot \hat{\mathbf{U}}^n_{i+1} - \boldsymbol{\varphi}(1) \cdot \hat{\mathbf{U}}^n_i \right]$$

1D Test Problems

Verification of the well-balanced property (*C*-property) [LeVeque 1998] using ***N*=20**, Δx=0.1):

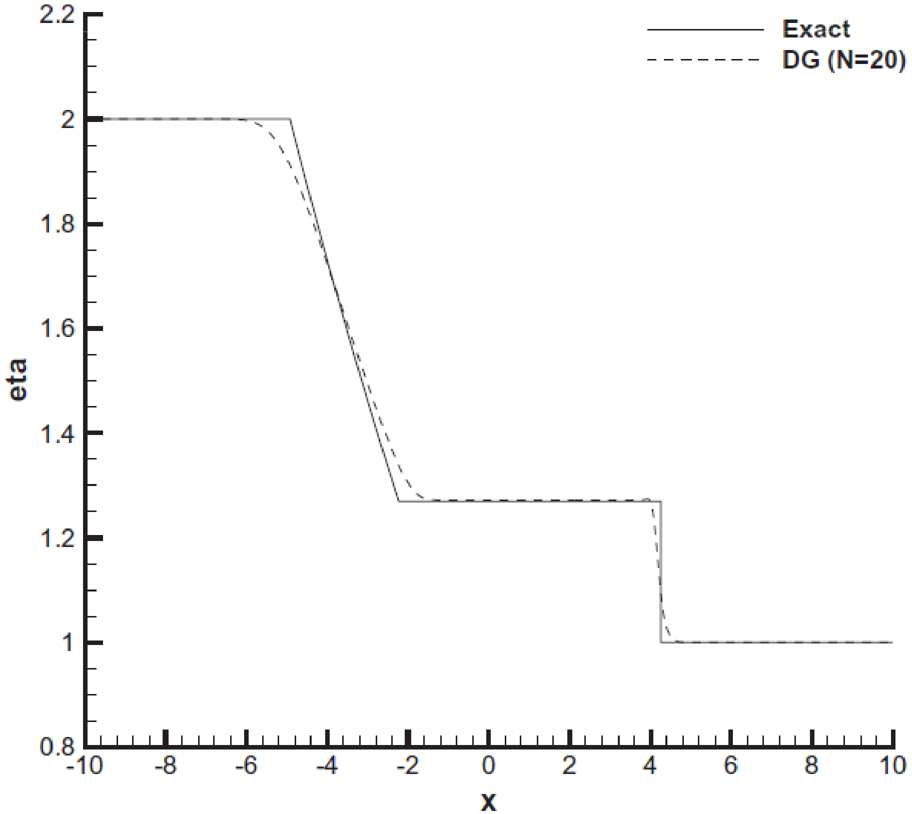


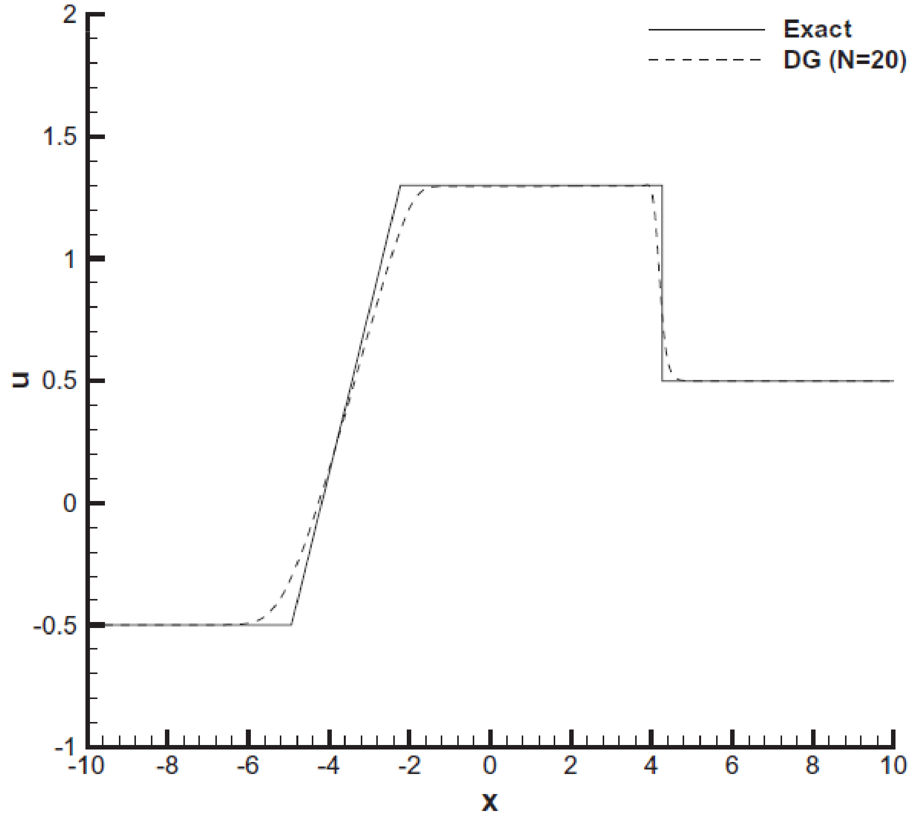


Case	L_∞ error	L_2 error
Single precision	4.2319298×10^{-5}	7.4470709×10^{-6}
Double precision	$2.1094237 \times 10^{-14}$	$4.9533439 \times 10^{-15}$
Quadruple precision	$9.6296497 \times 10^{-34}$	$5.3708826 \times 10^{-34}$

1D Test Problems

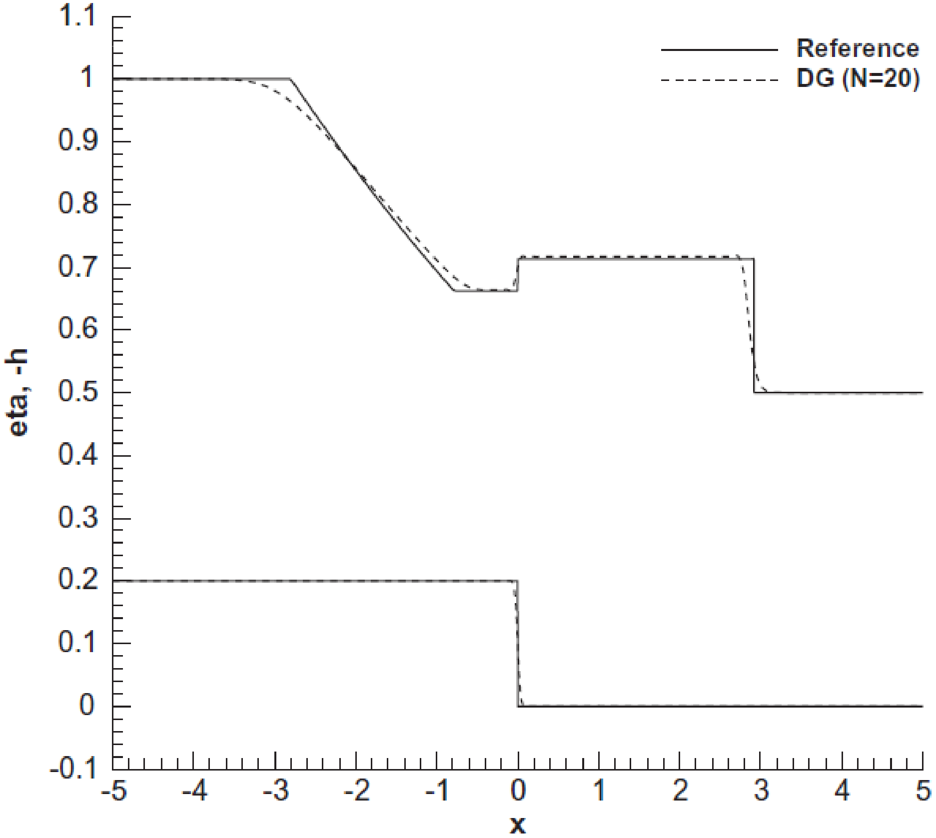
Solution of some Riemann problems (no limiters applied, initial discontinuity smoothed):

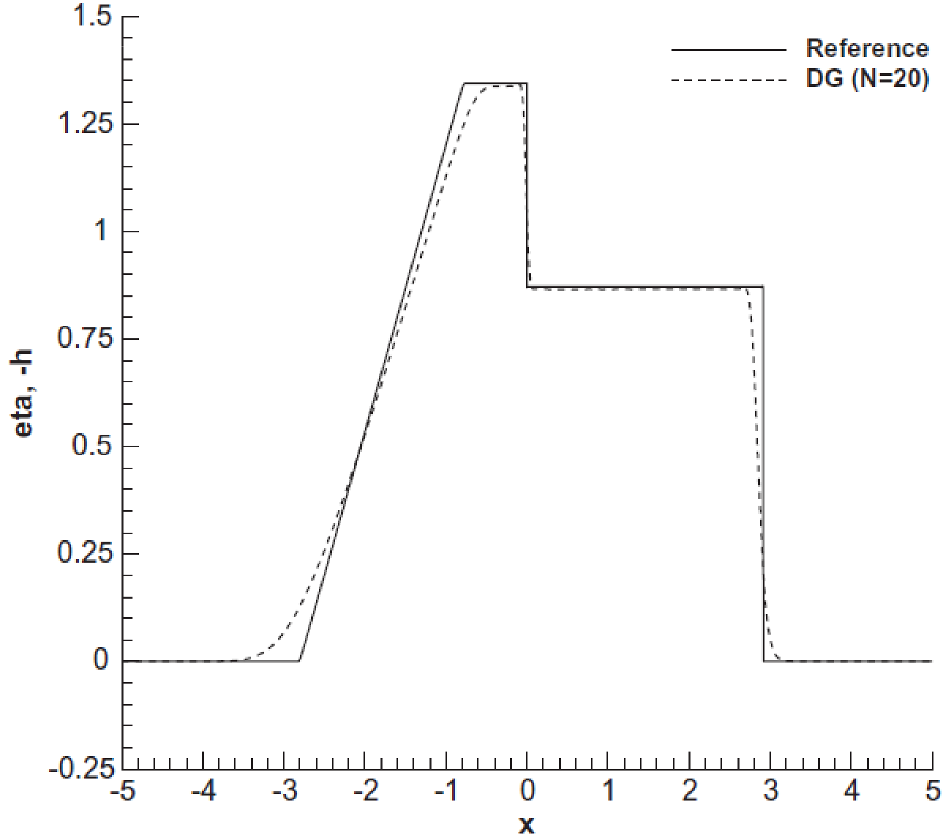




1D Test Problems

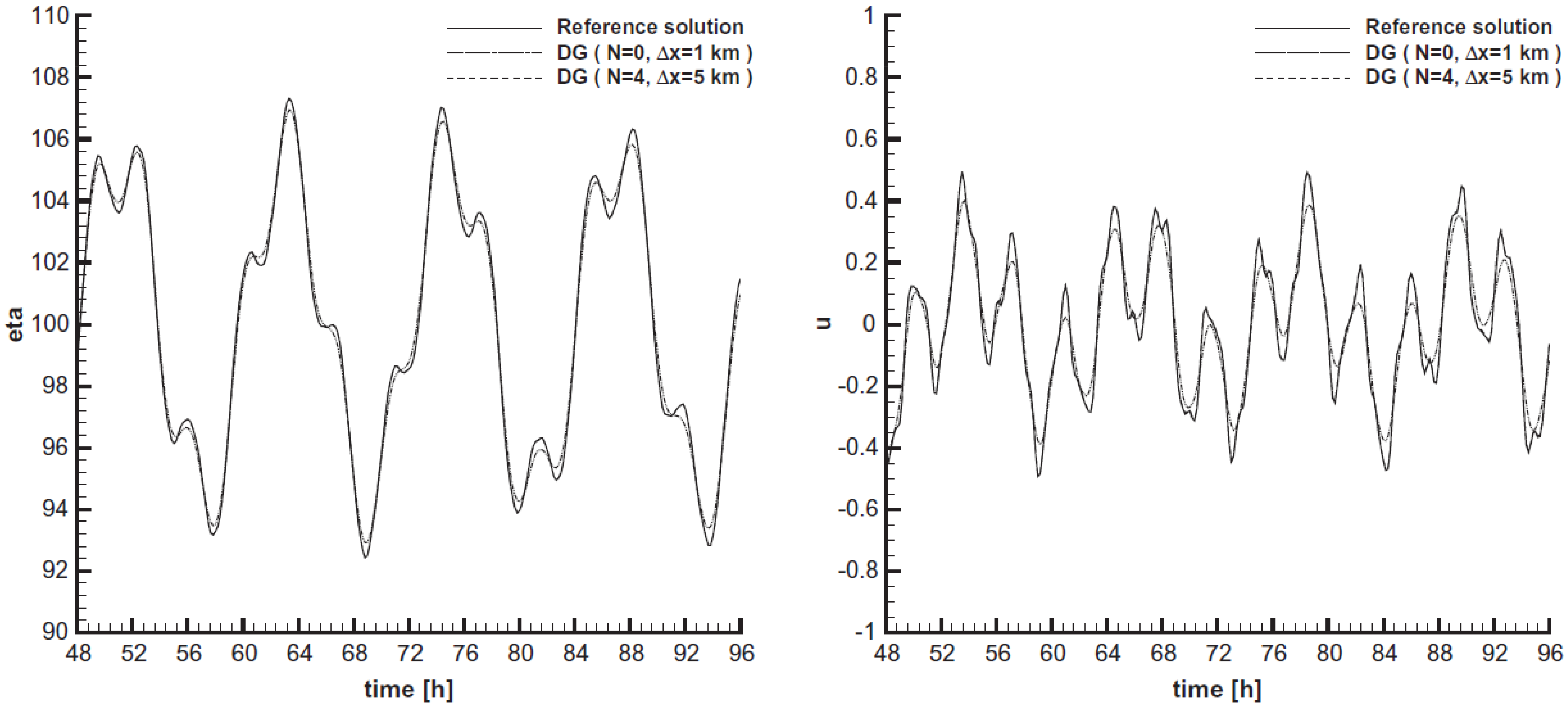
Solution of some Riemann problems (no limiters applied, initial discontinuity smoothed):





1D Test Problems

Low Froude number tidal-type flow.



Note that the higher order DG FEM solution matches the fine grid reference solution **perfectly well,** while the original *N=0* scheme does not, despite the same total number of degrees of freedom.

Extension to 2D

The 2D shallow water equations without sources and friction written in terms of conservative variables read:

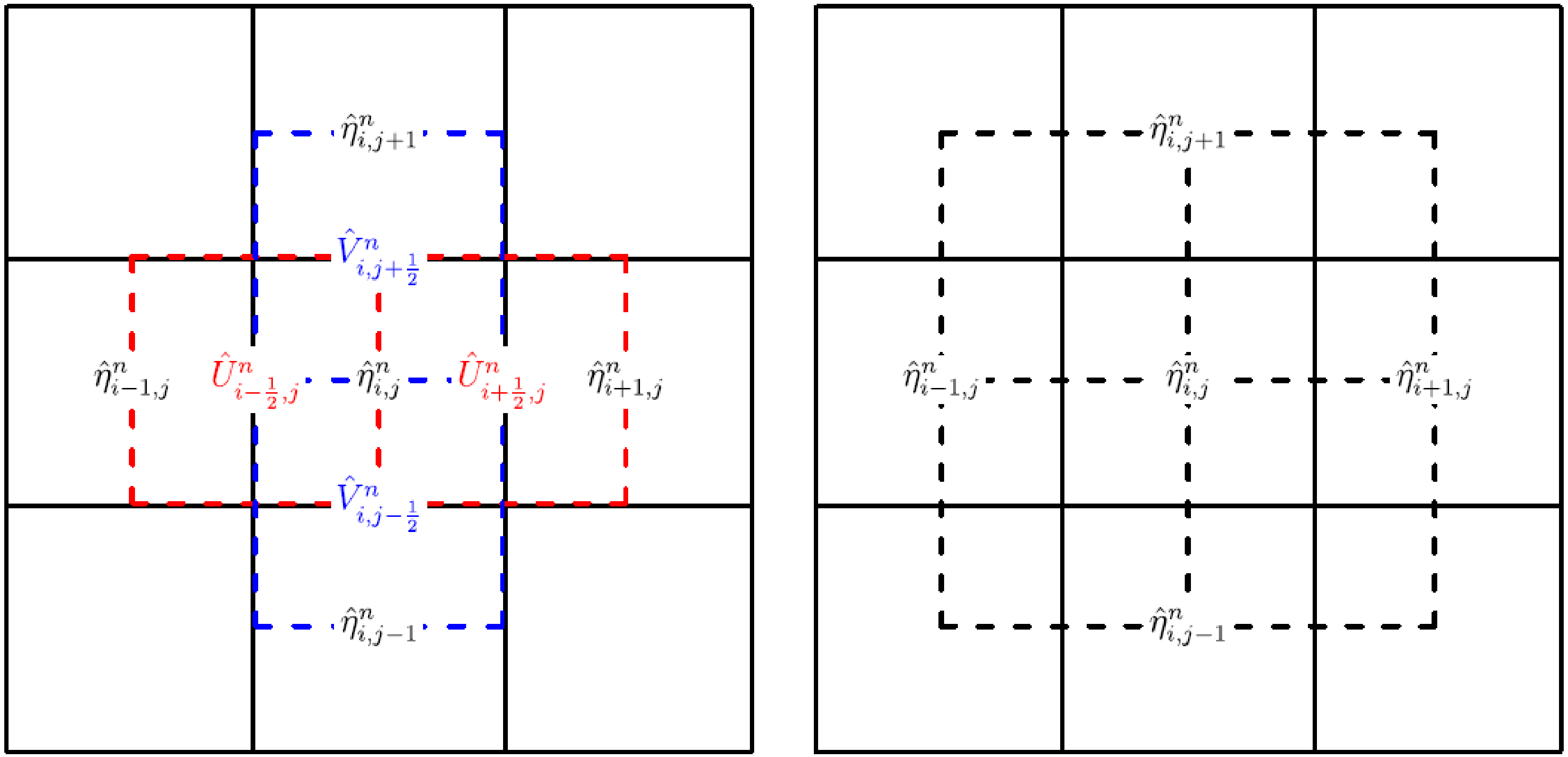
$$\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial vU}{\partial y} + gH\frac{\partial \eta}{\partial x} = 0,$$

$$\frac{\partial V}{\partial t} + \frac{\partial uV}{\partial x} + \frac{\partial vV}{\partial y} + gH\frac{\partial \eta}{\partial y} = 0,$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0,$$

Extension to 2D

Edge-based staggered grid. Free surface grid and velocity grids (left). For comparison the vertex-based staggered grid of [Liu, Shu, Tadmor and Zhang 2007, 2008]:



Our staggering corresponds exactly to the staggered grid used in TRIM.

Extension to 2D

Multiplication with test functions and integrating over the (staggered) control volumes:

$$\int_{T^u_{i+\frac{1}{2},j}} \phi \left(\frac{\partial U}{\partial t} + \frac{\partial uU}{\partial x} + \frac{\partial vU}{\partial y} + gH \frac{\partial \eta}{\partial x} \right) dy\, dx = 0,$$

$$\int_{T^v_{i,j+\frac{1}{2}}} \psi \left(\frac{\partial V}{\partial t} + \frac{\partial uV}{\partial x} + \frac{\partial vV}{\partial y} + gH \frac{\partial \eta}{\partial y} \right) dy\, dx = 0,$$

$$\int_{T^\eta_{i,j}} \omega \left(\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dy\, dx = 0.$$

$$\eta_{i,j}(x,y,t) = \omega(x,y) \cdot \hat{\boldsymbol{\eta}}_{i,j}(t)$$

$$U_{i+\frac{1}{2},j}(x,y,t) = \phi(x,y) \cdot \hat{\mathbf{U}}_{i+\frac{1}{2},j}(t), \quad H_{i+\frac{1}{2},j}(x,y,t) = \phi(x,y) \cdot \hat{\mathbf{H}}_{i+\frac{1}{2},j}(t)$$

$$V_{i,j+\frac{1}{2}}(x,y,t) = \psi(x,y) \cdot \hat{\mathbf{V}}_{i,j+\frac{1}{2}}(t), \quad H_{i,j+\frac{1}{2}}(x,y,t) = \psi(x,y) \cdot \hat{\mathbf{H}}_{i,j+\frac{1}{2}}(t)$$

Extension to 2D

Skipping the algebra (left for the coffee break) one gets

$$\mathbf{M} \cdot \left(\hat{\mathbf{U}}^{n+1}_{i+\frac{1}{2},j} - \widehat{\mathbf{FU}}^{n}_{i+\frac{1}{2},j} \right) + g \frac{\Delta t}{\Delta x} \left(\mathbf{R}^x_u \cdot \hat{\mathbf{H}}^n_{i+\frac{1}{2},j} \hat{\boldsymbol{\eta}}^{n+\theta}_{i+1,j} - \mathbf{L}^x_u \cdot \hat{\mathbf{H}}^n_{i+\frac{1}{2},j} \hat{\boldsymbol{\eta}}^{n+\theta}_{i,j} \right) = 0,$$

$$\mathbf{M} \cdot \left(\hat{\mathbf{V}}^{n+1}_{i,j+\frac{1}{2}} - \widehat{\mathbf{FV}}^{n}_{i,j+\frac{1}{2}} \right) + g \frac{\Delta t}{\Delta y} \left(\mathbf{R}^y_v \cdot \hat{\mathbf{H}}^n_{i,j+\frac{1}{2}} \hat{\boldsymbol{\eta}}^{n+\theta}_{i,j+1} - \mathbf{L}^y_v \cdot \hat{\mathbf{H}}^n_{i,j+\frac{1}{2}} \hat{\boldsymbol{\eta}}^{n+\theta}_{i,j} \right) = 0,$$

$$\mathbf{M} \cdot \left(\hat{\boldsymbol{\eta}}^{n+1}_{i,j} - \hat{\boldsymbol{\eta}}^{n}_{i,j} \right) + \frac{\Delta t}{\Delta x} \left(\mathbf{R}^x_\eta \cdot \hat{\mathbf{U}}^{n+\theta}_{i+\frac{1}{2},j} - \mathbf{L}^x_\eta \cdot \hat{\mathbf{U}}^{n+\theta}_{i-\frac{1}{2},j} \right) + \frac{\Delta t}{\Delta y} \left(\mathbf{R}^y_\eta \cdot \hat{\mathbf{V}}^{n+\theta}_{i,j+\frac{1}{2}} - \mathbf{L}^y_\eta \cdot \hat{\mathbf{V}}^{n+\theta}_{i,j-\frac{1}{2}} \right) = 0$$

$$\mathbf{M} = \int_0^1 \int_0^1 \boldsymbol{\varphi}(\xi,\gamma)\boldsymbol{\varphi}(\xi,\gamma) d\gamma\, d\xi,$$

$$\mathbf{L}^x_u = \int_0^1 \boldsymbol{\varphi}\left(\frac{1}{2},\gamma\right)\boldsymbol{\varphi}\left(\frac{1}{2},\gamma\right)\boldsymbol{\varphi}(1,\gamma) d\gamma - \int_0^{\frac{1}{2}} \int_0^1 \boldsymbol{\varphi}(\xi,\gamma)\boldsymbol{\varphi}(\xi,\gamma) \frac{\partial \boldsymbol{\varphi}(\xi+\frac{1}{2},\gamma)}{\partial \xi} d\gamma\, d\xi,$$

$$\mathbf{L}^y_v = \int_0^1 \boldsymbol{\varphi}\left(\xi,\frac{1}{2}\right)\boldsymbol{\varphi}\left(\xi,\frac{1}{2}\right)\boldsymbol{\varphi}(\xi,1) d\xi - \int_0^1 \int_0^{\frac{1}{2}} \boldsymbol{\varphi}(\xi,\gamma)\boldsymbol{\varphi}(\xi,\gamma) \frac{\partial \boldsymbol{\varphi}(\xi,\gamma+\frac{1}{2})}{\partial \gamma} d\gamma\, d\xi,$$

$$\mathbf{L}^x_\eta = \int_0^1 \boldsymbol{\varphi}(0,\gamma)\boldsymbol{\varphi}\left(\frac{1}{2},\gamma\right) d\gamma + \int_0^{\frac{1}{2}} \int_0^1 \frac{\partial \boldsymbol{\varphi}(\xi,\gamma)}{\partial \xi} \boldsymbol{\varphi}\left(\xi+\frac{1}{2},\gamma\right) d\gamma\, d\xi,$$

Extension to 2D

Inserting the discrete momentum equation into the discrete mass equation we get

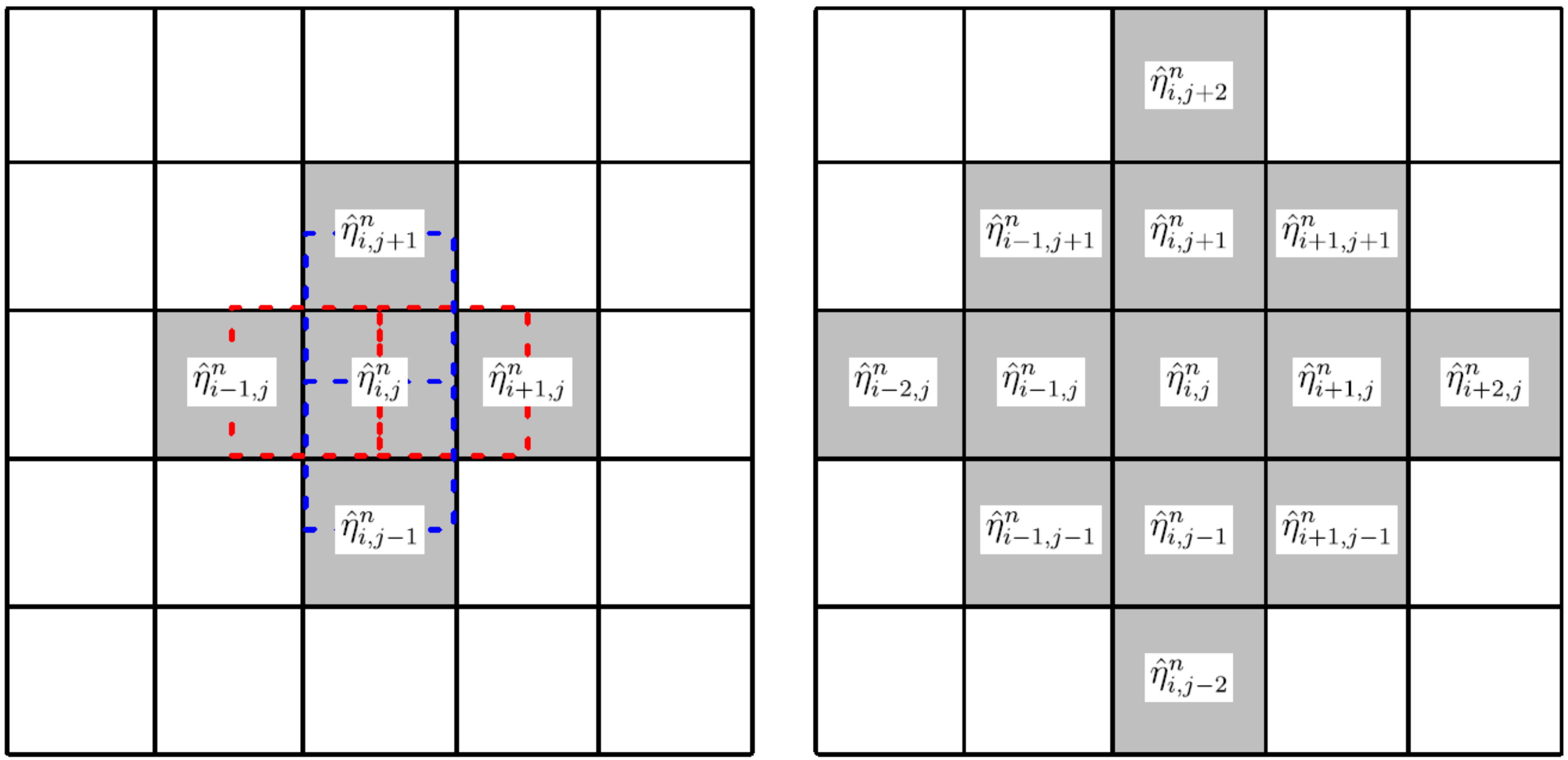
$$\mathbf{M}\cdot\hat{\boldsymbol{\eta}}_{i,j}^{n+1} - g\theta^2\frac{\Delta t^2}{\Delta x^2}\Big[\mathbf{R}_\eta^x\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_u^x\cdot\hat{\mathbf{H}}_{i+\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i+1,j}^{n+1} - \mathbf{L}_u^x\cdot\hat{\mathbf{H}}_{i+\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i,j}^{n+1}\Big) - \mathbf{L}_\eta^x\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_u^x\cdot\hat{\mathbf{H}}_{i-\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i,j}^{n+1} - \mathbf{L}_u^x\cdot\hat{\mathbf{H}}_{i-\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i-1,j}^{n+1}\Big)\Big]$$

$$- g\theta^2\frac{\Delta t^2}{\Delta y^2}\Big[\mathbf{R}_\eta^y\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_v^y\cdot\hat{\mathbf{H}}_{i,j+\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j+1}^{n+1} - \mathbf{L}_v^y\cdot\hat{\mathbf{H}}_{i,j+\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j}^{n+1}\Big) - \mathbf{L}_\eta^y\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_v^y\cdot\hat{\mathbf{H}}_{i,j-\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j}^{n+1} - \mathbf{L}_v^y\cdot\hat{\mathbf{H}}_{i,j-\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j-1}^{n+1}\Big)\Big] = \mathbf{b}_{i,j}^n,$$

$$\mathbf{b}_{i,j}^n = \mathbf{M}\cdot\hat{\boldsymbol{\eta}}_{i,j}^n - \frac{\Delta t}{\Delta x}\Big[\mathbf{R}_\eta^x\cdot\widehat{\mathbf{FU}}_{i+\frac{1}{2},j}^{n+\theta} - \mathbf{L}_\eta^x\cdot\widehat{\mathbf{FU}}_{i-\frac{1}{2},j}^{n+\theta}\Big] - \frac{\Delta t}{\Delta y}\Big[\mathbf{R}_\eta^y\cdot\widehat{\mathbf{FV}}_{i,j+\frac{1}{2}}^{n+\theta} - \mathbf{L}_\eta^y\cdot\widehat{\mathbf{FV}}_{i,j-\frac{1}{2}}^{n+\theta}\Big]$$
$$+ g\theta(1-\theta)\frac{\Delta t^2}{\Delta x^2}\Big[\mathbf{R}_\eta^x\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_u^x\cdot\hat{\mathbf{H}}_{i+\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i+1,j}^n - \mathbf{L}_u^x\cdot\hat{\mathbf{H}}_{i+\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i,j}^n\Big) - \mathbf{L}_\eta^x\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_u^x\cdot\hat{\mathbf{H}}_{i-\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i,j}^n - \mathbf{L}_u^x\cdot\hat{\mathbf{H}}_{i-\frac{1}{2},j}^n\hat{\boldsymbol{\eta}}_{i-1,j}^n\Big)\Big]$$
$$+ g\theta(1-\theta)\frac{\Delta t^2}{\Delta y^2}\Big[\mathbf{R}_\eta^y\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_v^y\cdot\hat{\mathbf{H}}_{i,j+\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j+1}^n - \mathbf{L}_v^y\cdot\hat{\mathbf{H}}_{i,j+\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j}^n\Big) - \mathbf{L}_\eta^y\cdot\mathbf{M}^{-1}\cdot\Big(\mathbf{R}_v^y\cdot\hat{\mathbf{H}}_{i,j-\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j}^n - \mathbf{L}_v^y\cdot\hat{\mathbf{H}}_{i,j-\frac{1}{2}}^n\hat{\boldsymbol{\eta}}_{i,j-1}^n\Big)\Big]$$

Extension to 2D

In the special case *N=0* the **staggered DG** method reduces to the original TRIM scheme of [Casulli 1990], which is **not the case** for the **collocated** semi-implicit DG FEM scheme of [Tumolo, Bonaventura, Restelli 2013]. The present staggered DG FEM scheme uses a **five-point stencil** for the wave equation in 2D, while the TBR scheme uses a **13 point** stencil for the wave equation!



Convergence test

Stationary vortex: free surface gradient balanced by centrifugal force

$$\frac{\partial \eta}{\partial r} = \frac{u_\alpha^2}{gr}.$$

$$\eta(x,y,0) = 1 - \frac{1}{2g} e^{-(r^2-1)}, \quad u(x,y,0) = -u_\alpha \sin(\alpha),$$

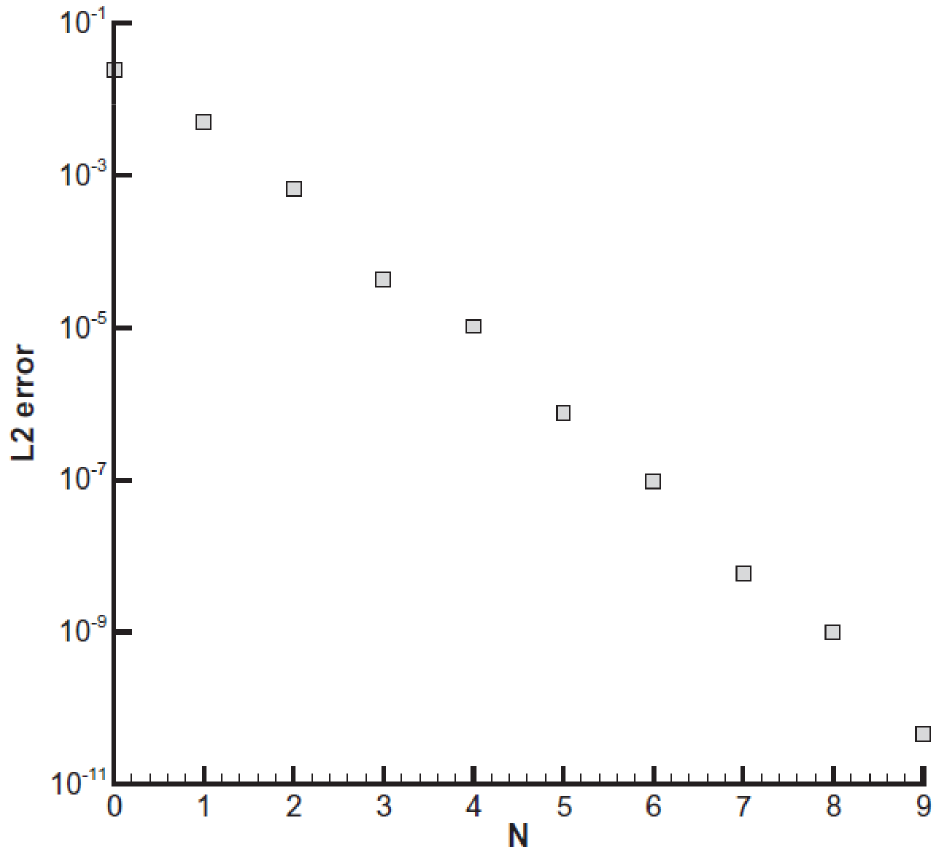
$$u_\alpha(r,0) = r e^{-\frac{1}{2}(r^2-1)} \qquad v(x,y,0) = u_\alpha \cos(\alpha)$$

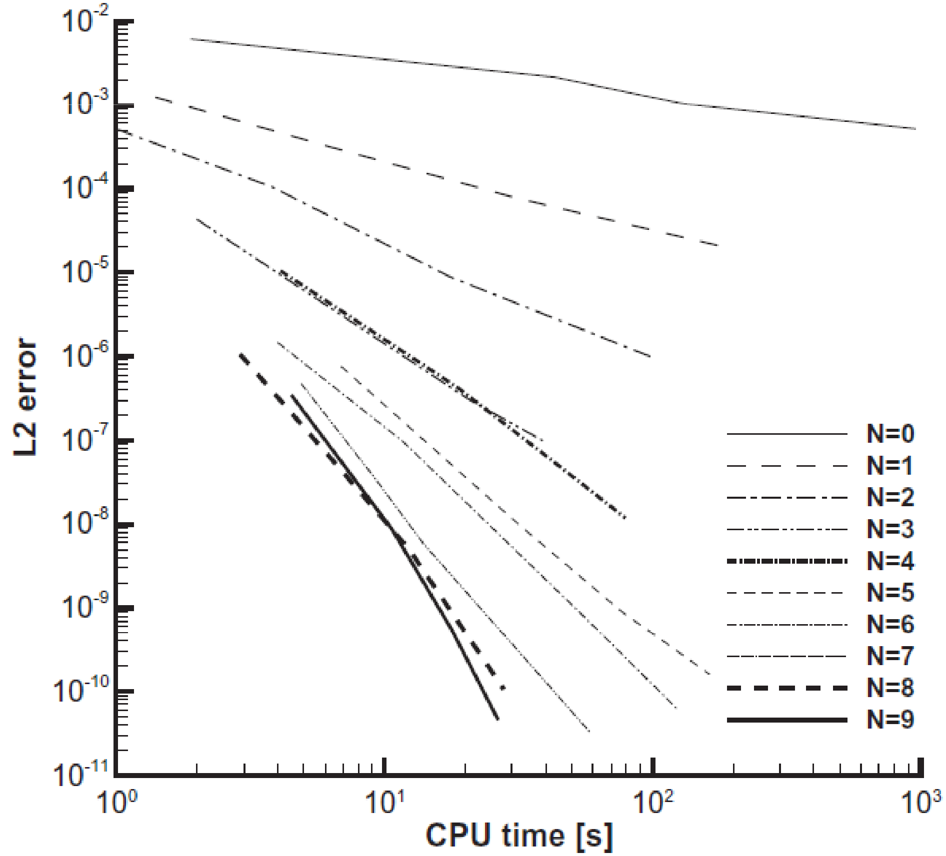
http://en.kllproject.lv/2011/07/19/

http://cgcookie.com/blender/images/maelstrom-w-i-p-2/

N_x	ϵ_{L_2}	$\mathcal{O}(L_2)$	$t_{CPU}[s]$	N_x	ϵ_{L_2}	$\mathcal{O}(L_2)$	$t_{CPU}[s]$
$N=1$				$N=2$			
50	1.2348E−03		1.4	25	6.7018E−04		0.8
100	3.0909E−04	2.0	6.4	50	1.0514E−04	2.7	3.8
200	7.6496E−05	2.0	31.4	100	8.8111E−06	3.6	17.7
400	1.8867E−05	2.0	202.6	200	9.5849E−07	3.2	101.2
$N=3$				$N=4$			
25	4.2970E−05		2.0	25	1.0488E−05		4.1
50	1.8188E−06	4.6	8.9	50	4.2312E−07	4.6	18.8
75	2.9992E−07	4.4	20.9	75	5.4965E−08	5.0	43.0
100	9.5429E−08	4.0	39.8	100	1.1860E−08	5.3	79.3
$N=5$				$N=6$			
25	7.5968E−07		6.9	15	1.4658E−06		4.0
50	1.0972E−08	6.1	30.6	25	9.5381E−08	5.3	11.8
75	9.0768E−10	6.1	76.7	50	8.7960E−10	6.8	53.7
100	1.6179E−10	6.0	163.2	75	6.1558E−11	6.6	124.1
$N=7$				$N=8$			
15	4.7500E−07		4.9	10	1.0609E−06		2.9
20	3.7408E−08	8.8	9.1	15	4.6751E−08	7.7	6.8
25	5.8485E−09	8.3	14.0	20	4.9707E−09	7.8	12.4
50	3.2941E−11	7.5	58.5	30	1.0677E−10	9.5	27.9
$N=9$				$N=0$			
10	3.4647E−07		4.5	100	6.0804E−03		1.9
15	8.6376E−09	9.1	10.7	250	2.1422E−03	1.1	43.2
20	5.5780E−10	9.5	17.8	500	1.0348E−03	1.0	129.8
25	4.5964E−11	11.2	26.6	1000	5.1822E−04	1.0	957.0

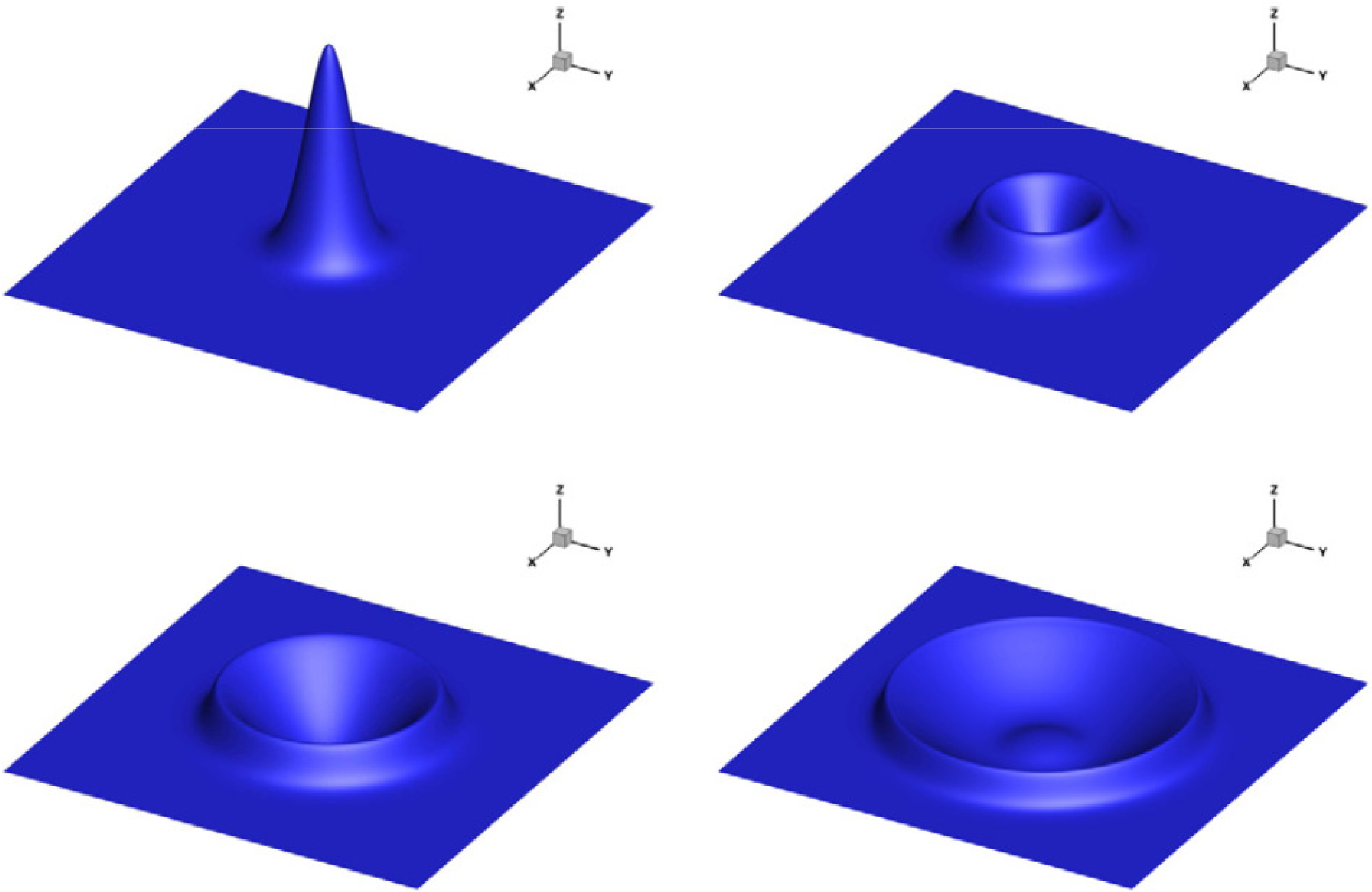
Convergence test



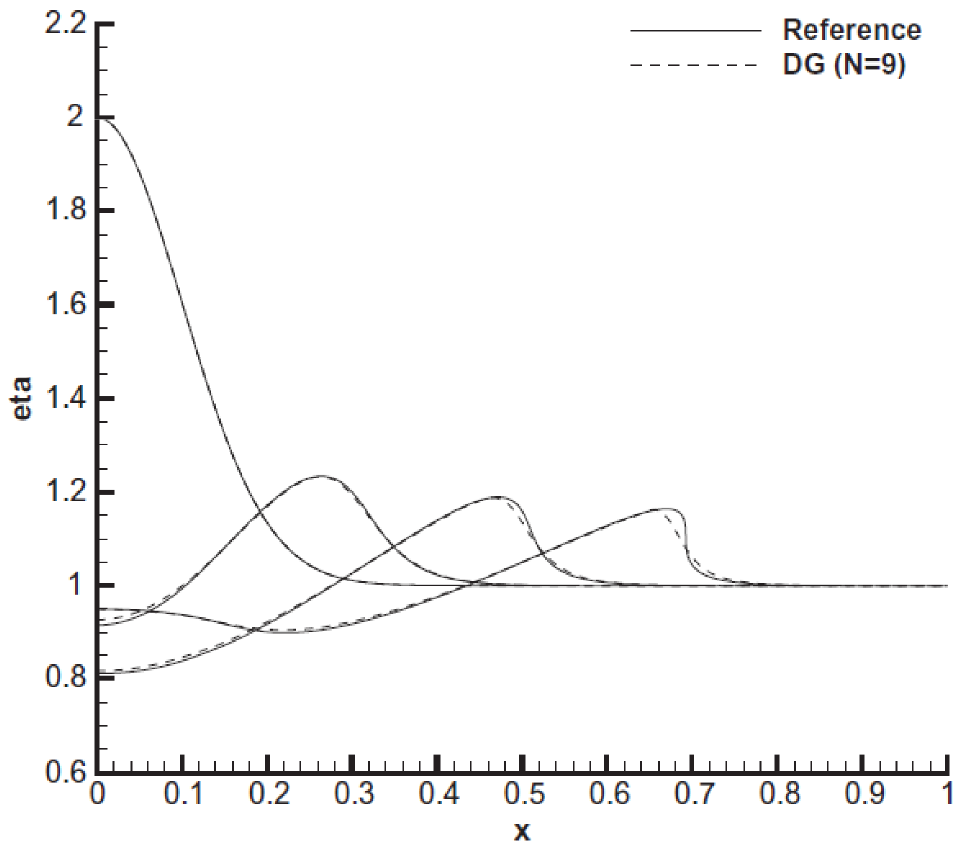


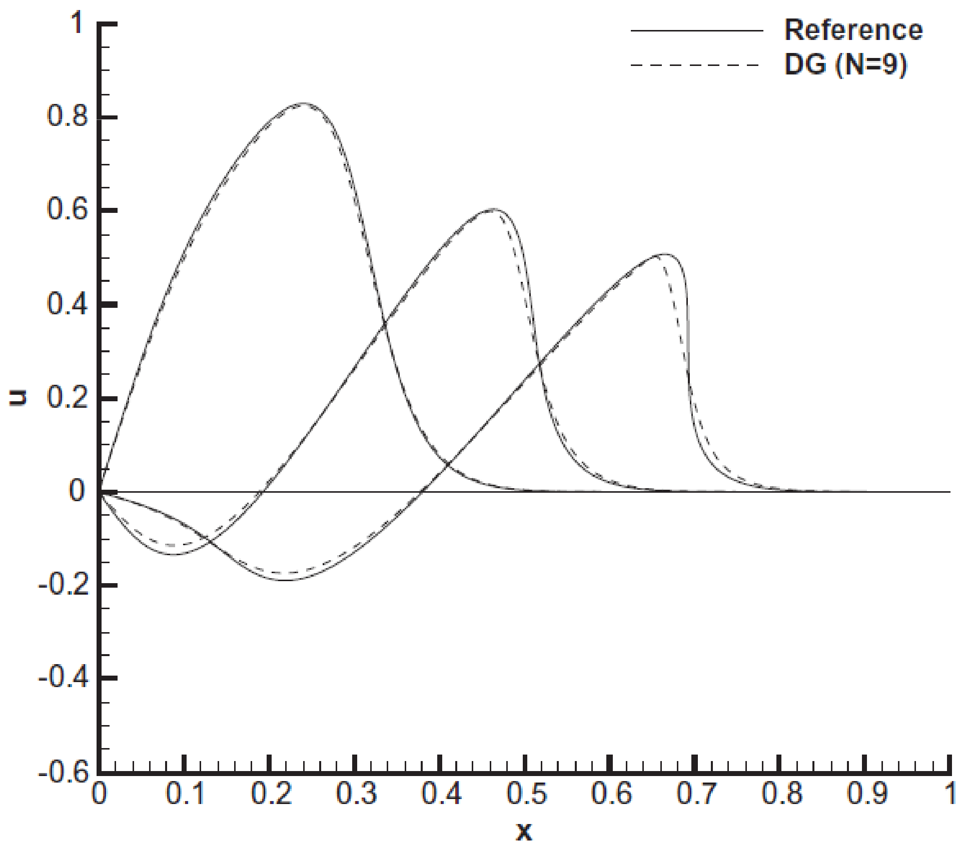
$$\epsilon_{L_2} = \int_{\Omega} (\eta_h - \eta_e)^2 \, dy \, dx,$$

Smooth 2D Wave Propagation

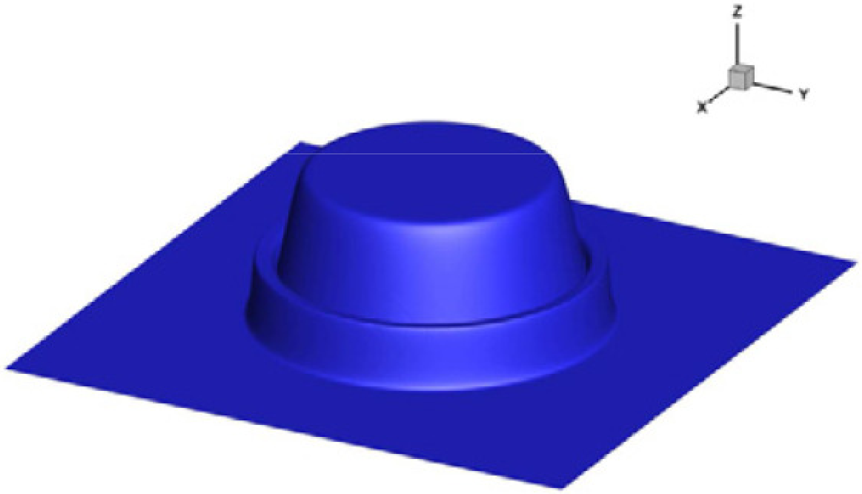


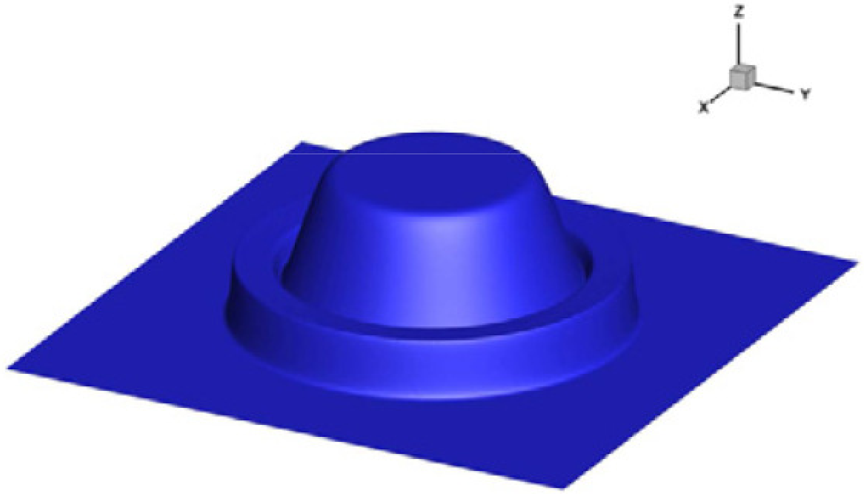
Smooth 2D wave propagation

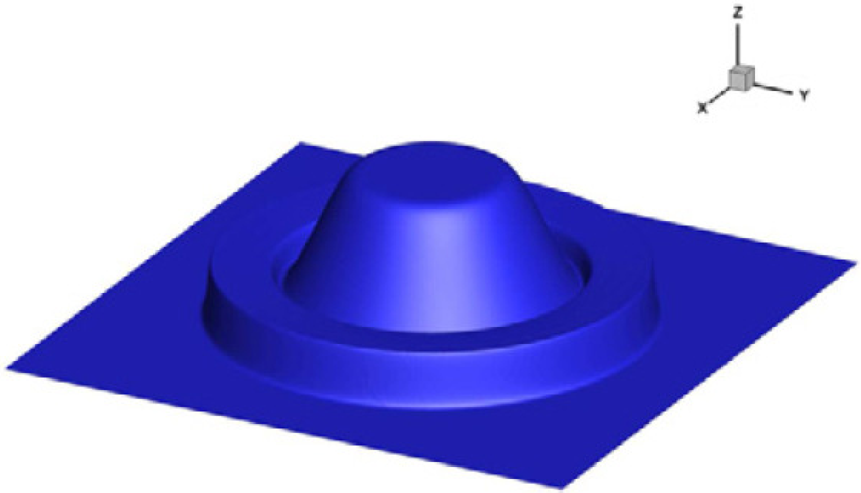


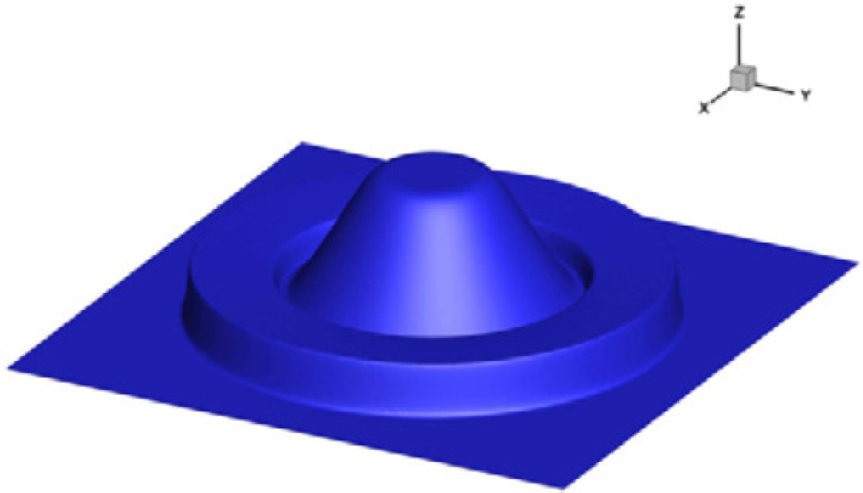


2D Circular Dam Break

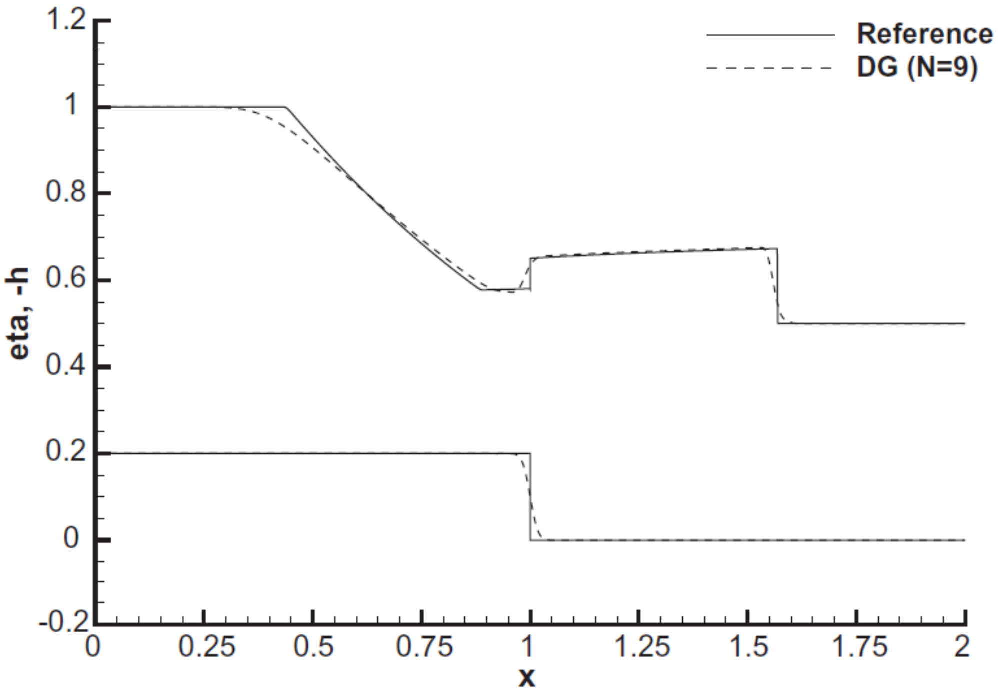


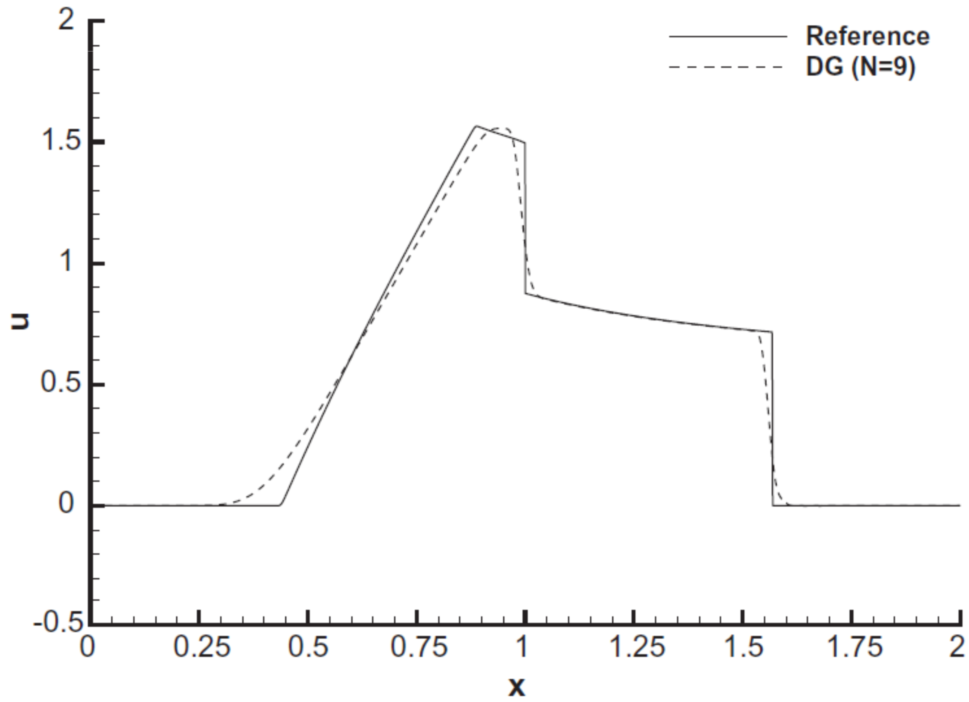






2D Circular Dam Break





3D Incompressible Navier-Stokes equations

The momentum equation and the mass conservation equation read

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F}_c + \nabla p = \nabla \cdot (\nu \nabla \mathbf{v}) + \mathbf{S},$$

$$\nabla \cdot \mathbf{v} = 0,$$

with the flux tensor of the nonlinear convective terms

$$\mathbf{F}_c = \begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix}$$

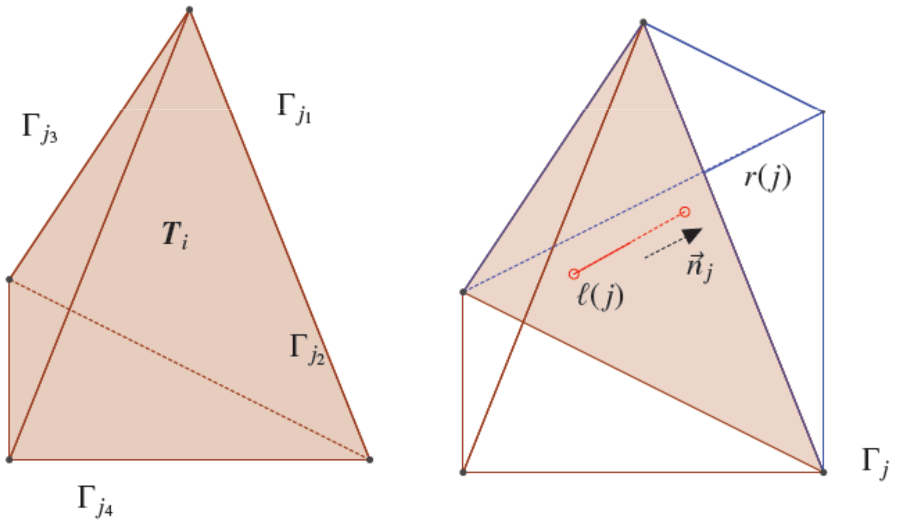
The momentum equation can also be rewritten more compactly as

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F} + \nabla p = \mathbf{S}$$

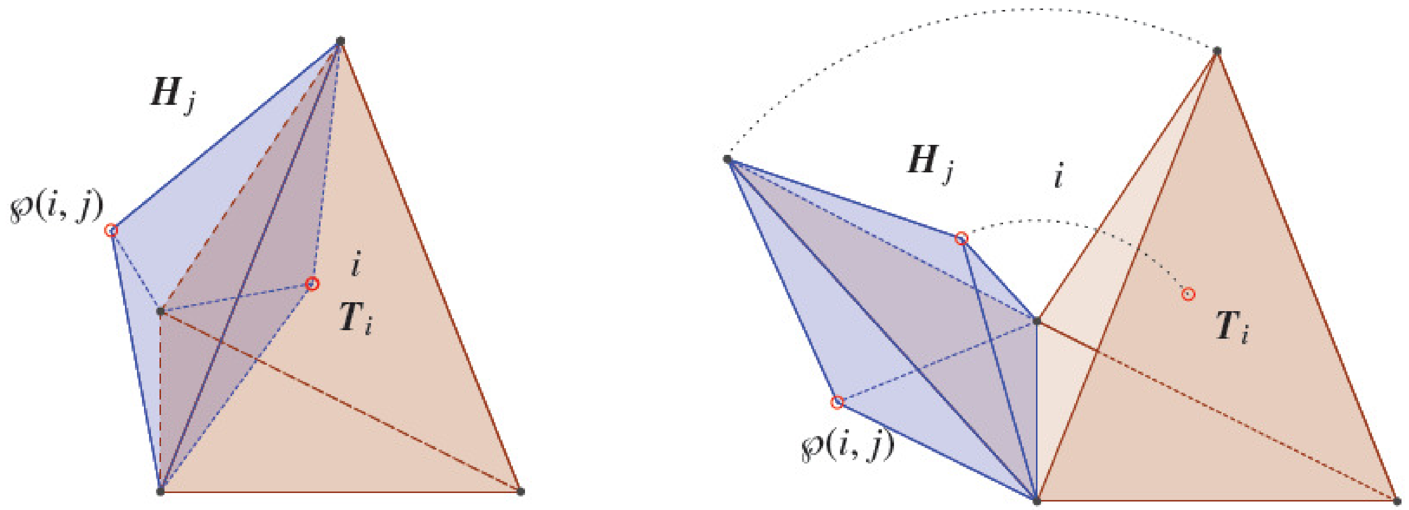
$$\mathbf{F} = \mathbf{F}(\mathbf{v}, \nabla \mathbf{v}) = \mathbf{F}_c(\mathbf{v}) - \nu \nabla \mathbf{v}$$

3D Incompressible Navier-Stokes equations

Unstructured tetrahedral grid and notation for faces, normal vectors and left/right elements



Face-based dual element (non-standard 5-point hexahedron, highlighted in blue)



3D Incompressible Navier-Stokes equations

Discrete solution is expanded in terms of space-time basis functions on the primary and the face-based staggered dual mesh:

$$p_i(\mathbf{x},t) = \sum_{l=1}^{N_\phi^{st}} \tilde{\phi}_l^{(i)}(\mathbf{x},t)\hat{p}_{l,i}^{n+1} =: \tilde{\boldsymbol{\phi}}^{(i)}(\mathbf{x},t)\hat{\boldsymbol{p}}_i^{n+1},$$

$$\mathbf{v}_j(\mathbf{x},t) = \sum_{l=1}^{N_\psi^{st}} \tilde{\psi}_l^{(j)}(\mathbf{x},t)\hat{\mathbf{v}}_{l,j}^{n+1} =: \tilde{\boldsymbol{\psi}}^{(j)}(\mathbf{x},t)\hat{\mathbf{v}}_j^{n+1},$$

Derivation of a weak formulation of the mass and momentum equation, respectively:

$$\int_{\boldsymbol{T}_i^{st}} \tilde{\phi}_k^{(i)} \nabla \cdot \mathbf{v}\, d\mathbf{x}dt = 0,$$

$$\int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \left(\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F} \right) d\mathbf{x}dt + \int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \nabla p \; d\mathbf{x}dt = \int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \mathbf{S}\, d\mathbf{x}dt,$$

3D Incompressible Navier-Stokes equations

Integration by parts of the continuity equation yields:

$$\sum_{j \in S_i} \left(\int_{\Gamma_j^{st}} \tilde{\phi}_k^{(i)} \mathbf{v}_j \cdot \vec{n}_{ij} \, dS \, dt - \int_{\boldsymbol{T}_{i,j}^{st}} \nabla \tilde{\phi}_k^{(i)} \cdot \mathbf{v}_j \, d\mathbf{x} dt \right) = 0,$$

The pressure jump along the face Γ_j inside the dual element can be easily interpreted in the sense of distributions (or alternatively as Bassi-Rebay lifting operator), hence:

$$\int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \left(\frac{\partial \mathbf{v}_j}{\partial t} + \nabla \cdot \mathbf{F} \right) d\mathbf{x} dt + \int_{\boldsymbol{T}_{\ell(j),j}^{st}} \tilde{\psi}_k^{(j)} \nabla p_{\ell(j)} \, d\mathbf{x} dt + \int_{\boldsymbol{T}_{r(j),j}^{st}} \tilde{\psi}_k^{(j)} \nabla p_{r(j)} \, d\mathbf{x} dt + \int_{\Gamma_j^{st}} \tilde{\psi}_k^{(j)} \left(p_{r(j)} - p_{\ell(j)} \right) \vec{n}_j \, dS \, dt = \int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \mathbf{S} \, d\mathbf{x} dt,$$

Integration of the first term by parts in time yields:

$$\int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \frac{\partial \mathbf{v}_j}{\partial t} \, d\mathbf{x} dt = \int_{\boldsymbol{H}_j} \tilde{\psi}_k^{(j)}(\mathbf{x}, t^{n+1}) \mathbf{v}_j(\mathbf{x}, t^{n+1}) \, d\mathbf{x} - \int_{\boldsymbol{H}_j} \tilde{\psi}_k^{(j)}(\mathbf{x}, t^n) \mathbf{v}_j(\mathbf{x}, t^n) \, d\mathbf{x} - \int_{\boldsymbol{H}_j^{st}} \frac{\partial \tilde{\psi}_k^{(j)}}{\partial t} \mathbf{v}_j(\mathbf{x}, t) \, d\mathbf{x} dt.$$

3D Incompressible Navier-Stokes equations

In a more compact matrix-vector notation, we get the following scheme:

$$\sum_{j \in S_i} \mathcal{D}_{i,j} \hat{\mathbf{v}}_j^{n+1} = 0,$$

$$\boldsymbol{M}_j \hat{\mathbf{v}}_j^{n+1} - \boldsymbol{M}_j \widehat{\boldsymbol{F}\mathbf{v}}_j + \boldsymbol{Q}_{r(j),j} \hat{\boldsymbol{p}}_{r(j)}^{n+1} + \boldsymbol{Q}_{\ell(j),j} \hat{\boldsymbol{p}}_{\ell(j)}^{n+1} = 0,$$

As usual in semi-implicit schemes, we insert the discrete momentum equation into the discrete continuity equation, to get one single equation for the discrete pressure

$$\sum_{j \in S_i} \mathcal{D}_{i,j} \boldsymbol{M}_j^{-1} \boldsymbol{Q}_{i,j} \hat{\boldsymbol{p}}_i^{n+1} + \sum_{j \in S_i} \mathcal{D}_{i,j} \boldsymbol{M}_j^{-1} \boldsymbol{Q}_{\wp(i,j),j} \hat{\boldsymbol{p}}_{\wp(i,j)}^{n+1} = \sum_{j \in S_i} \mathcal{D}_{i,j} \widehat{\boldsymbol{F}\mathbf{v}}_j.$$

Note: on a *collocated* grid, the same algorithm would either produce a **17-point stencil** (instead of the **5-point stencil** on the staggered grid), or it would lead to a system with **four times** more unknowns (pressure *p* **and** all velocity components u, v, w, if a 5-point stencil is used).

3D Incompressible Navier-Stokes equations

Nonlinear convective terms (explicit) and viscous terms (implicit), via operator splitting and an outer Picard iteration, to maintain high order in time. The convective terms are discretized on the main grid (requires averaging from the dual to the primary grid), and the viscous terms are discretized again on the staggered dual grid by defining the viscous stress tensor on the dual grid.

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot \mathbf{F}_c + \nabla \cdot \sigma = 0,$$
$$\sigma = -\nu \nabla \mathbf{v}.$$

The averaging operators from the dual to the main grid and vice versa read

$$\bar{\mathbf{v}}_i^{n+1} = \boldsymbol{M}_i^{-1} \sum_{j \in S_i} \boldsymbol{M}_{i,j} \hat{\mathbf{v}}_j^{n+1} \qquad \boldsymbol{M}_i = \int_{\boldsymbol{T}_i^{st}} \tilde{\phi}_k^{(i)} \tilde{\phi}_l^{(i)} d\mathbf{x}dt, \qquad \boldsymbol{M}_{i,j} = \int_{\boldsymbol{T}_{i,j}^{st}} \tilde{\phi}_k^{(i)} \tilde{\psi}_l^{(j)} d\mathbf{x}dt$$

$$\hat{\mathbf{v}}_j^{n+1} = \overline{\boldsymbol{M}}_j^{-1} \left(\boldsymbol{M}_{\ell(j),j}^{\top} \bar{\mathbf{v}}_{\ell(j)}^{n+1} + \boldsymbol{M}_{r(j),j}^{\top} \bar{\mathbf{v}}_{r(j)}^{n+1} \right) \qquad \overline{\boldsymbol{M}}_j = \int_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)} \tilde{\psi}_l^{(j)} d\mathbf{x}dt$$

By adding and subtracting the pressure gradient at the old Picard iteration, one can derive an efficient and high order accurate space-time pressure correction algorithm.

3D Incompressible Navier-Stokes equations

The weak form of the momentum equation then reads

$$\int\limits_{\boldsymbol{T}_i} \tilde{\phi}_k^{(i)}(\mathbf{x}, t^{n+1})\bar{\mathbf{v}}_i^{n+1}\, d\mathbf{x} - \int\limits_{\boldsymbol{T}_i} \tilde{\phi}_k^{(i)}(\mathbf{x}, t^{n})\bar{\mathbf{v}}_i^{n}\, d\mathbf{x} - \int\limits_{\boldsymbol{T}_i^{st}} \frac{\partial \tilde{\phi}_k^{(i)}}{\partial t}\bar{\mathbf{v}}_i^{n+1}\, d\mathbf{x}dt +$$

$$\int\limits_{\partial \boldsymbol{T}_i^{st}} \tilde{\phi}_k^{(i)}\mathbf{F}_c^{\mathrm{RS}}\left(\bar{\mathbf{v}}^-, \bar{\mathbf{v}}^+\right)\cdot \vec{n}_i\, dSdt - \int\limits_{\boldsymbol{T}_i^{st}} \nabla\tilde{\phi}_k^{(i)}\cdot \mathbf{F}_c(\bar{\mathbf{v}}_i^{n+1})\, d\mathbf{x}dt + \sum_{j\in S_i}\left(\int\limits_{\Gamma_j^{st}} \tilde{\phi}_k^{(i)}\boldsymbol{\sigma}_j^{n+1}\cdot\vec{n}_{ij}\, dSdt - \int\limits_{\boldsymbol{T}_{i,j}^{st}} \nabla\tilde{\phi}_k^{(i)}\cdot\boldsymbol{\sigma}_j^{n+1}\, d\mathbf{x}dt\right) = 0, \tag{40}$$

$$\int\limits_{\boldsymbol{H}_j^{st}} \tilde{\psi}_k^{(j)}(\mathbf{x}, t^{n+1})\boldsymbol{\sigma}_j^{n+1}\, d\mathbf{x}$$

$$= -\nu\left(\int\limits_{\boldsymbol{T}_{\ell(j),j}^{st}} \tilde{\psi}_k^{(j)}\nabla\bar{\mathbf{v}}_{\ell(j)}^{n+1}\, d\mathbf{x}dt + \int\limits_{\boldsymbol{T}_{r(j),j}^{st}} \tilde{\psi}_k^{(j)}\nabla\bar{\mathbf{v}}_{r(j)}^{n+1}\, d\mathbf{x}dt + \int\limits_{\Gamma_j^{st}} \tilde{\psi}_k^{(j)}\left(\bar{\mathbf{v}}_{r(j)}^{n+1} - \bar{\mathbf{v}}_{\ell(j)}^{n+1}\right)\otimes\vec{n}_j\, dSdt\right). \tag{41}$$

or, more compactly,

$$\left(\overline{\boldsymbol{M}}_i^+ - \overline{\boldsymbol{M}}_i^0\right)\bar{\mathbf{v}}_i^{n+1} - \overline{\boldsymbol{M}}_i^-\bar{\mathbf{v}}_i^n + \sum_{j\in S_i}\mathcal{D}_{i,j}\boldsymbol{\sigma}_j^{n+1} + \overline{\Upsilon}_i^c = 0,$$

$$\overline{\boldsymbol{M}}_j\boldsymbol{\sigma}_j^{n+1} = -\nu\left(\mathcal{Q}_{\ell(j),j}\bar{\mathbf{v}}_{\ell(j)}^{n+1} + \mathcal{Q}_{r(j),j}\bar{\mathbf{v}}_{r(j)}^{n+1}\right)$$

The final scheme reads

$$\bar{\mathbf{v}}_i^{n+1,k} = \boldsymbol{M}_i^{-1} \sum_{j \in S_i} \boldsymbol{M}_{i,j} \hat{\mathbf{v}}_j^{n+1,k},$$

$$\boldsymbol{\Lambda}_i(\hat{\boldsymbol{p}}^{n+1,k}) = \boldsymbol{M}_i^{-1} \sum_{j \in S_i} \boldsymbol{M}_{i,j} \left(\boldsymbol{M}_j^{-1} \left(\boldsymbol{Q}_{r(j),j} \hat{\boldsymbol{p}}_{r(j)}^{n+1,k} + \boldsymbol{Q}_{\ell(j),j} \hat{\boldsymbol{p}}_{\ell(j)}^{n+1,k} \right) \right),$$

$$\overline{\Upsilon}_i^c(\bar{\mathbf{v}}) = \int_{\partial \boldsymbol{T}_i^{st}} \tilde{\phi}_k^{(i)} \mathbf{F}_c^{\mathrm{RS}} \left(\bar{\mathbf{v}}^-, \bar{\mathbf{v}}^+ \right) \cdot \vec{n}_i \, dS \, dt - \int_{\boldsymbol{T}_i^{st}} \nabla \tilde{\phi}_k^{(i)} \cdot \mathbf{F}_c(\bar{\mathbf{v}}) \, d\mathbf{x} dt,$$

$$\left(\overline{\boldsymbol{M}}_i - \nu \sum_{j \in S_i} \mathcal{D}_{i,j} \overline{\boldsymbol{M}}_j^{-1} \boldsymbol{Q}_{i,j} \right) \overline{\mathbf{v}}_i^{n+1,k+\frac{1}{2}} - \nu \sum_{j \in S_i} \mathcal{D}_{i,j} \overline{\boldsymbol{M}}_j^{-1} \boldsymbol{Q}_{\wp(i,j),j} \overline{\mathbf{v}}_{\wp(i,j)}^{n+1,k+\frac{1}{2}} = \overline{\boldsymbol{M}}_i^{-} \overline{\mathbf{v}}_i^n - \overline{\Upsilon}_i^c \left(\overline{\mathbf{v}}^{n+1,k} \right) - \overline{\boldsymbol{M}}_i \boldsymbol{\Lambda}_i(\hat{\boldsymbol{p}}^{n+1,k}),$$

$$\widehat{\boldsymbol{F}\mathbf{v}}_j^{n+1,k+\frac{1}{2}} = \overline{\boldsymbol{M}}_j^{-1} \left(\boldsymbol{M}_{\ell(j),j}^{\top} \overline{\mathbf{v}}_{\ell(j)}^{n+1,k+\frac{1}{2}} + \boldsymbol{M}_{r(j),j}^{\top} \overline{\mathbf{v}}_{r(j)}^{n+1,k+\frac{1}{2}} \right),$$

$$\sum_{j \in S_i} \mathcal{D}_{i,j} \boldsymbol{M}_j^{-1} \boldsymbol{Q}_{i,j} \left(\hat{\boldsymbol{p}}_i^{n+1,k+1} - \hat{\boldsymbol{p}}_i^{n+1,k} \right) + \sum_{j \in S_i} \mathcal{D}_{i,j} \boldsymbol{M}_j^{-1} \boldsymbol{Q}_{\wp(i,j),j} \left(\hat{\boldsymbol{p}}_{\wp(i,j)}^{n+1,k+1} - \hat{\boldsymbol{p}}_{\wp(i,j)}^{n+1,k} \right) = \sum_{j \in S_i} \mathcal{D}_{i,j} \widehat{\boldsymbol{F}\mathbf{v}}_j^{n+1,k+\frac{1}{2}},$$

$$\hat{\mathbf{v}}_j^{n+1,k+1} = \widehat{\boldsymbol{F}\mathbf{v}}_j^{n+1,k+\frac{1}{2}} - \boldsymbol{M}_j^{-1} \left(\boldsymbol{Q}_{r(j),j} \left(\hat{\boldsymbol{p}}_{r(j)}^{n+1,k+1} - \hat{\boldsymbol{p}}_{r(j)}^{n+1,k} \right) + \boldsymbol{Q}_{\ell(j),j} \left(\hat{\boldsymbol{p}}_{\ell(j)}^{n+1,k+1} - \hat{\boldsymbol{p}}_{\ell(j)}^{n+1,k} \right) \right),$$

Properties of the Linear Systems

As already mentioned before, our new staggered DG scheme has

- **optimal compactness** (stencil for all systems involves only the cell and direct neighbors)
- Minimum number of unknowns (only **scalar problems** need to be solved)

For piecewise constant polynomials in time, one can **prove** that

- the viscous system is **symmetric** and **positive definite**
- the pressure system is **symmetric** and **positive definite** (for appropriate BC)
- the CG method applied to the solution of the linear systems **without any preconditioner** converges with optimal rate in $s^{1/d}$ iterations, where s is the problem size (number of elements times number of DOF/element) and d is the number of space dimensions.

For high order polynomials in time, one can **prove** that

- the pressure system can still be written in a **symmetric** and **positive definite** form (for appropriate BC)
- in general, the viscous system is **no longer symmetric**, but it can be written as a ν perturbation of the identity matrix, which is very well conditioned for low viscosities.

L2 Stability Proof

Suppose the viscous and convective terms **F** are **directly discretized** with a DG scheme on the **dual grid** (e.g. using an LDG method), unlike the method presented before. Further, assume the DG discretization of **F** to be L2 stable (use of monotone e-fluxes in the convective terms and LDG for the viscous terms). Then the following theorem holds:

Theorem: The semi-discrete staggered DG scheme is L2 stable in the sense

$$\frac{d}{dt}\int_{\Omega}\frac{1}{2}\left(u_h^2+v_h^2\right)d\mathbf{x}\leqslant 0.$$

Proof: By construction, the weak solution of the staggered DG scheme satisfies

$$\int_{\boldsymbol{R}_j}\psi_k^{(j)}\frac{\partial}{\partial t}\mathbf{v}_h\,d\mathbf{x}+\int_{\partial\boldsymbol{R}_j}\psi_k^{(j)}\mathbf{G}_h\cdot\vec{n}\,ds-\int_{\boldsymbol{R}_j}\nabla\psi_k^{(j)}\cdot\mathbf{F}(\mathbf{v}_h,\nabla\mathbf{v}_h)\,d\mathbf{x}$$
$$+\int_{\boldsymbol{T}_{\ell(j),j}}\psi_k^{(j)}\nabla p_{h,\ell(j)}\,d\mathbf{x}+\int_{\boldsymbol{T}_{r(j),j}}\psi_k^{(j)}\nabla p_{h,r(j)}\,d\mathbf{x}+\int_{\Gamma_j}\psi_k^{(j)}\left(p_{h,r(j)}-p_{h,\ell(j)}\right)\vec{n}_{std}\,ds=0$$

$$\sum_{j\in S_i}\left[\int_{\Gamma_j}\phi_k^{(i)}\mathbf{v}_h\cdot\vec{n}_{e,j}\,ds-\int_{\boldsymbol{T}_{i,j}}\nabla\phi_k^{(i)}\cdot\mathbf{v}_h\,d\mathbf{x}\right]=0$$

L2 Stability Proof

Inserting the discrete pressure p_h as test function in the weak form of the continuity equation and inserting the velocity $\mathbf{v}_h$ as test function in the weak momentum equation and summing over all components yields

$$\sum_{j \in S_i} \left[\int_{\Gamma_j} p_{h,i} \, \mathbf{v}_h \cdot \vec{n}_{e,j} \, ds - \int_{\boldsymbol{T}_{i,j}} \nabla p_{h,i} \cdot \mathbf{v}_h \, d\mathbf{x} \right] = 0.$$

$$\int_{\boldsymbol{R}_j} \mathbf{v}_h \cdot \frac{\partial}{\partial t} \mathbf{v}_h \, d\mathbf{x} + \int_{\partial \boldsymbol{R}_j} \mathbf{v}_h \cdot (\mathbf{G}_h \cdot \vec{n}) \, ds - \int_{\boldsymbol{R}_j} \nabla \mathbf{v}_h : \mathbf{F}(\mathbf{v}_h, \nabla \mathbf{v}_h) \, d\mathbf{x}$$
$$+ \int_{\boldsymbol{T}_{\ell(j),j}} \mathbf{v}_h \cdot \nabla p_{h,\ell(j)} \, d\mathbf{x} + \int_{\boldsymbol{T}_{r(j),j}} \mathbf{v}_h \cdot \nabla p_{h,r(j)} \, d\mathbf{x} + \int_{\Gamma_j} \left(p_{h,r(j)} - p_{h,\ell(j)} \right) \mathbf{v}_h \cdot \vec{n}_{std} \, ds = 0$$

with the operator $c = \mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$

Summing the first equation over all elements yields

$$\sum_{i=1}^{N_i} \sum_{j \in S_i} \left[\int_{\Gamma_j} p_{h,i} \mathbf{v}_h \cdot \vec{n}_{e,j} \, ds - \int_{\boldsymbol{T}_{i,j}} \nabla p_{h,i} \cdot \mathbf{v}_h \right] d\mathbf{x} = 0,$$

$$\sum_{j=1}^{N_j} \left[\int_{\boldsymbol{T}_{\ell(j),j}} \nabla p_{h,\ell(j)} \cdot \mathbf{v}_h \, d\mathbf{x} + \int_{\boldsymbol{T}_{r(j),j}} \nabla p_{h,r(j)} \cdot \mathbf{v}_h \, d\mathbf{x} + \int_{\Gamma_j} \left(p_{h,r(j)} - p_{h,\ell(j)} \right) \mathbf{v}_h \cdot \vec{n}_{std} \, ds \right] = 0$$

L2 Stability Proof

From the weak form of the momentum equation, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_h^2+v_h^2)\,d\mathbf{x} = \int_\Omega \left(\mathbf{v}_h\cdot\frac{\partial}{\partial t}\mathbf{v}_h\right)d\mathbf{x} = \sum_{j=1}^{N_j}\int_{\mathbf{R}_j}\left(\mathbf{v}_h\cdot\frac{\partial}{\partial t}\mathbf{v}_h\right)d\mathbf{x}$$

$$= \sum_{j=1}^{N_j}\left[-\int_{\partial\mathbf{R}_j}\mathbf{v}_h\cdot(\mathbf{G}_h\cdot\vec{n})\,ds + \int_{\mathbf{R}_j}\nabla\mathbf{v}_h:\mathbf{F}(\mathbf{v}_h,\nabla\mathbf{v}_h)\,d\mathbf{x}\right]$$

$$-\sum_{j=1}^{N_j}\left[\int_{\mathbf{T}_{\ell(j),j}}\mathbf{v}_h\cdot\nabla p_{h,\ell(j)}\,d\mathbf{x} + \int_{\mathbf{T}_{r(j),j}}\mathbf{v}_h\cdot\nabla p_{h,r(j)}\,d\mathbf{x} + \int_{\Gamma_j}\left(p_{h,r(j)}-p_{h,\ell(j)}\right)\mathbf{v}_h\cdot\vec{n}_{std}\,ds\right]$$

The last row vanishes, thanks to the weak form of the continuity equation. We thus obtain

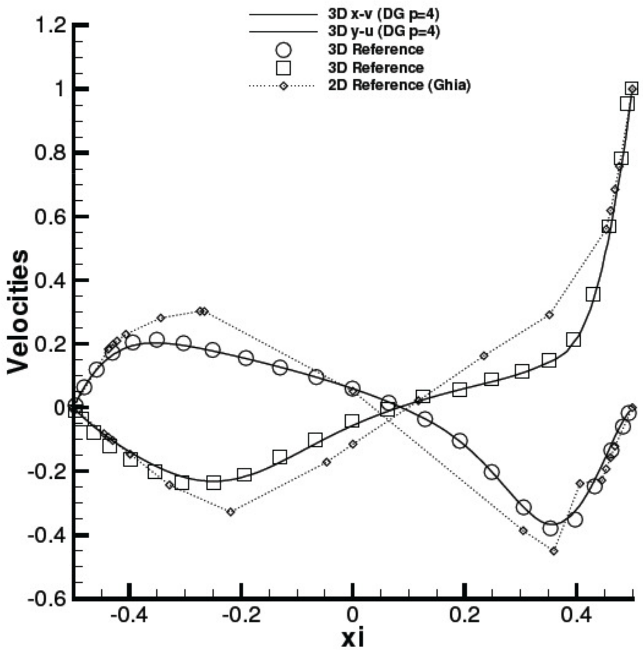
$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_h^2+v_h^2)d\mathbf{x} = -\sum_{j=1}^{N_j}\left[\int_{\partial\mathbf{R}_j}\mathbf{v}_h\cdot(\mathbf{G}_h\cdot\vec{n})\,ds - \int_{\mathbf{R}_j}\nabla\mathbf{v}_h:\mathbf{F}(\mathbf{v}_h,\nabla\mathbf{v}_h)\,d\mathbf{x}\right]$$

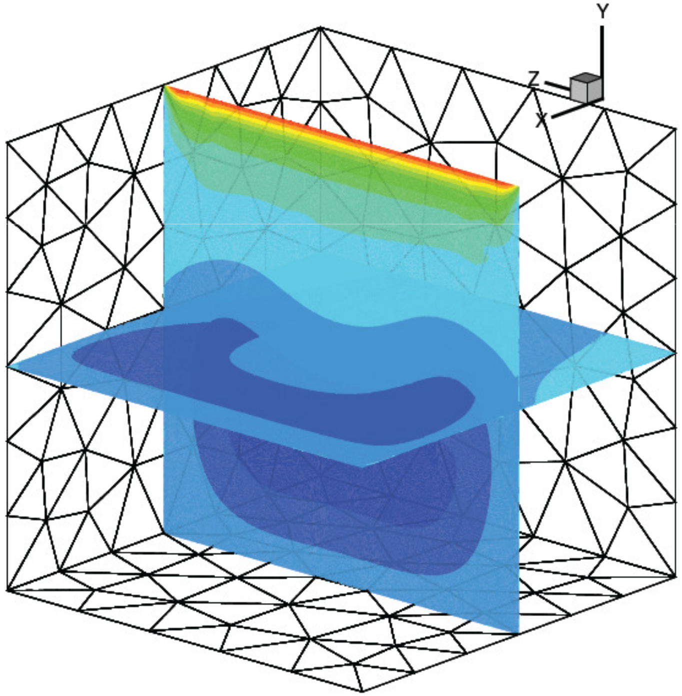
which is a standard DG scheme for **F** on the dual mesh. Since we suppose this DG scheme to be stable, we get

$$\frac{d}{dt}\int_\Omega \frac{1}{2}(u_h^2+v_h^2)\,d\mathbf{x} \leqslant 0.$$

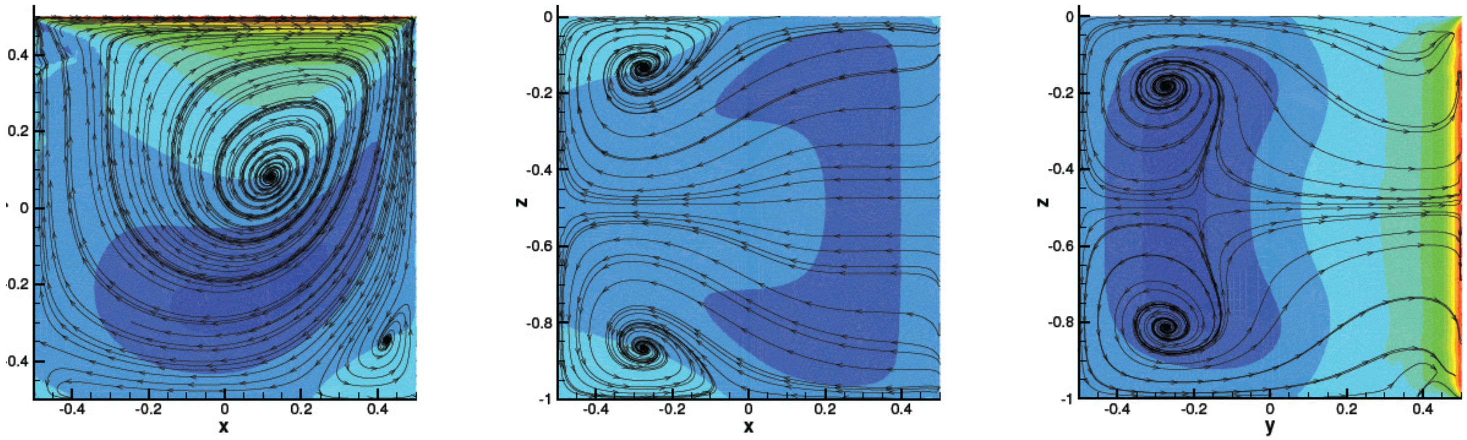
It is obvious that for **F**=0, our staggered DG scheme is **kinetic energy preserving**.

3D Lid-Driven Cavity

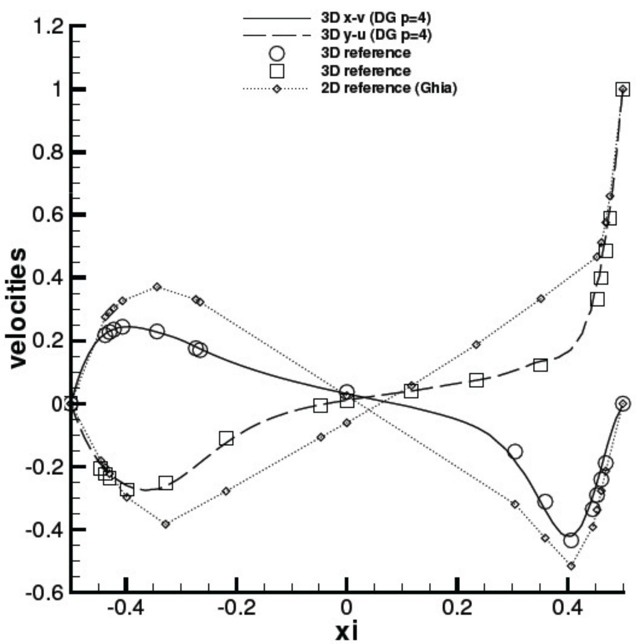


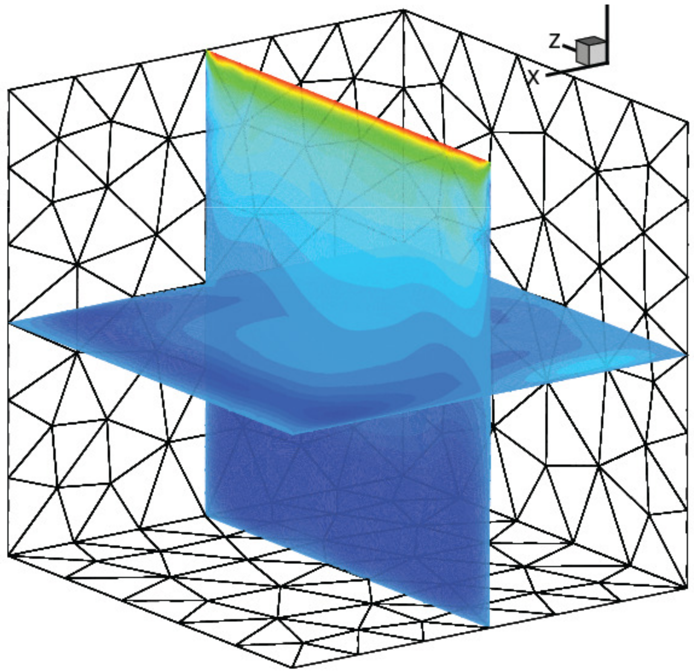


Re = 400

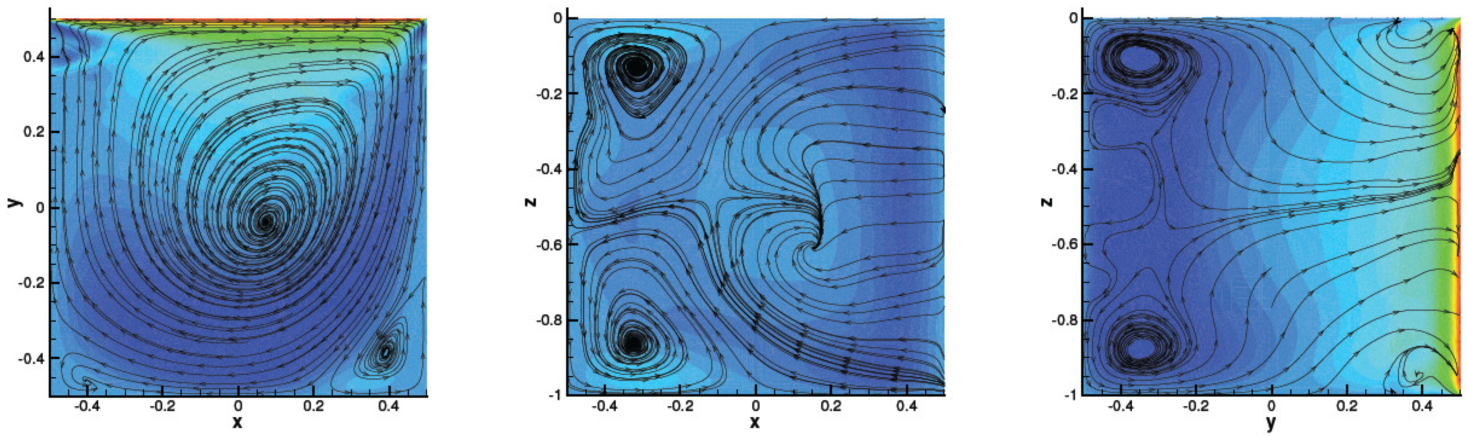


3D Lid-Driven Cavity

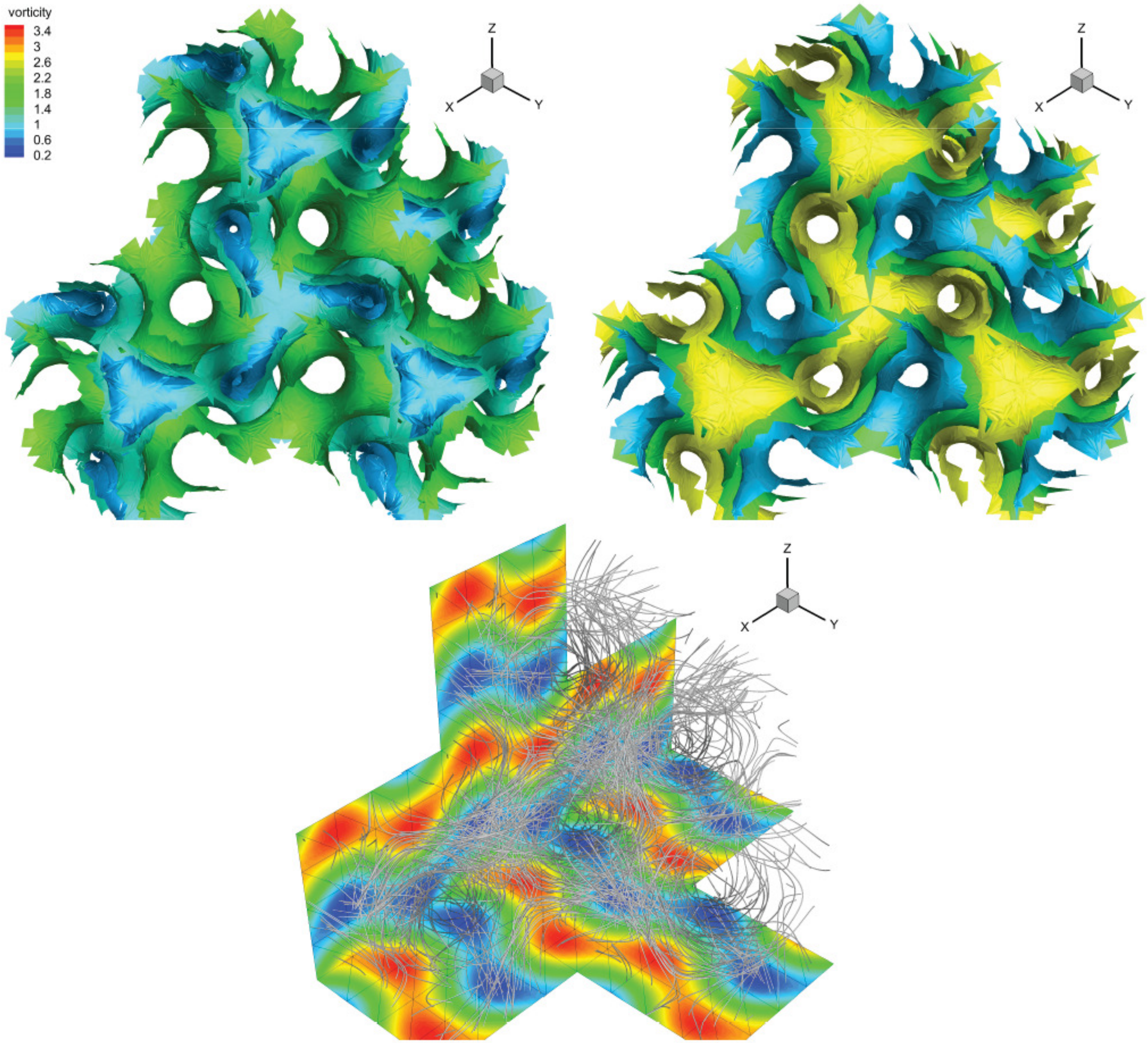




Re = 1000



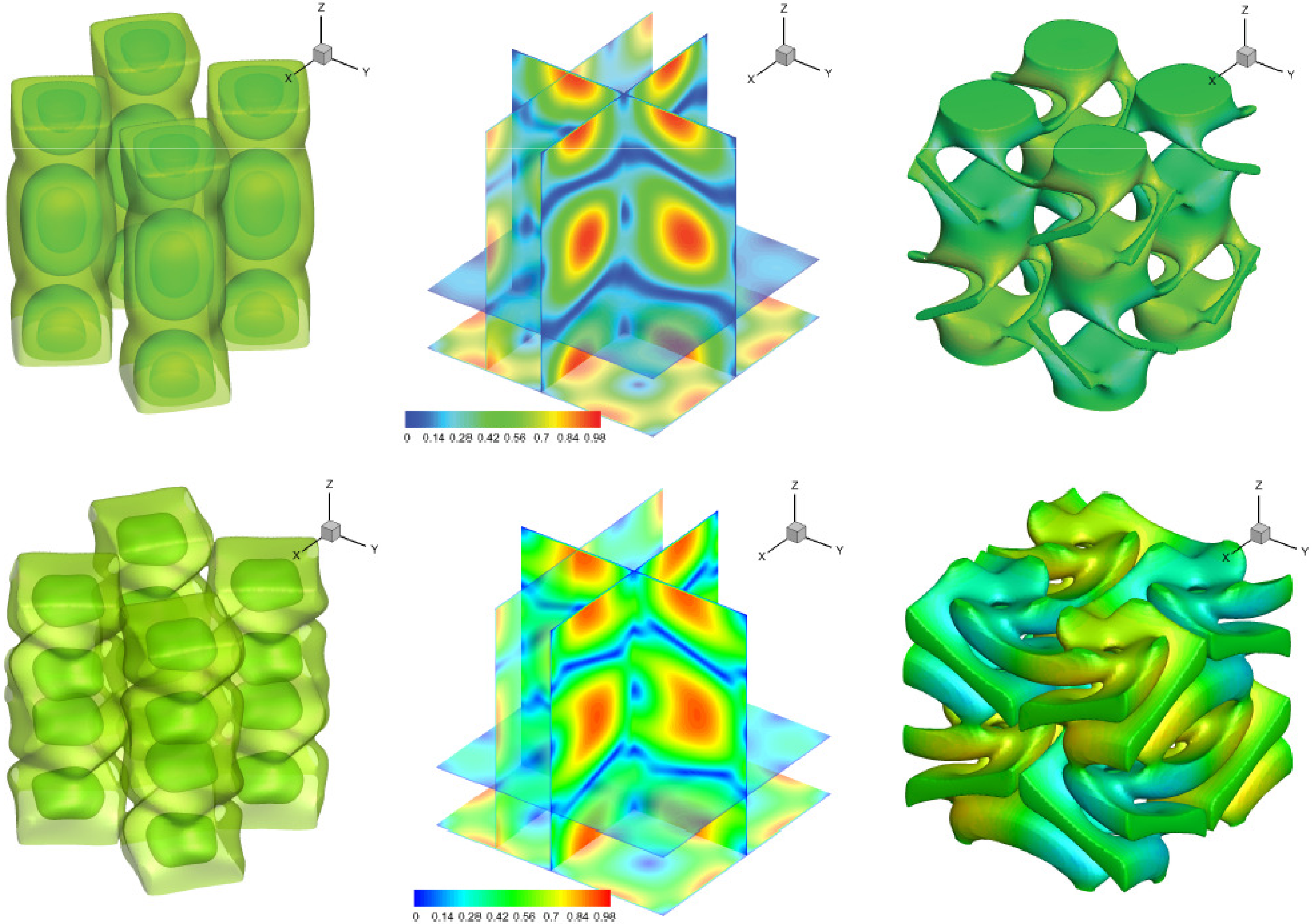
3D Arnold-Beltrami-Childress Flow



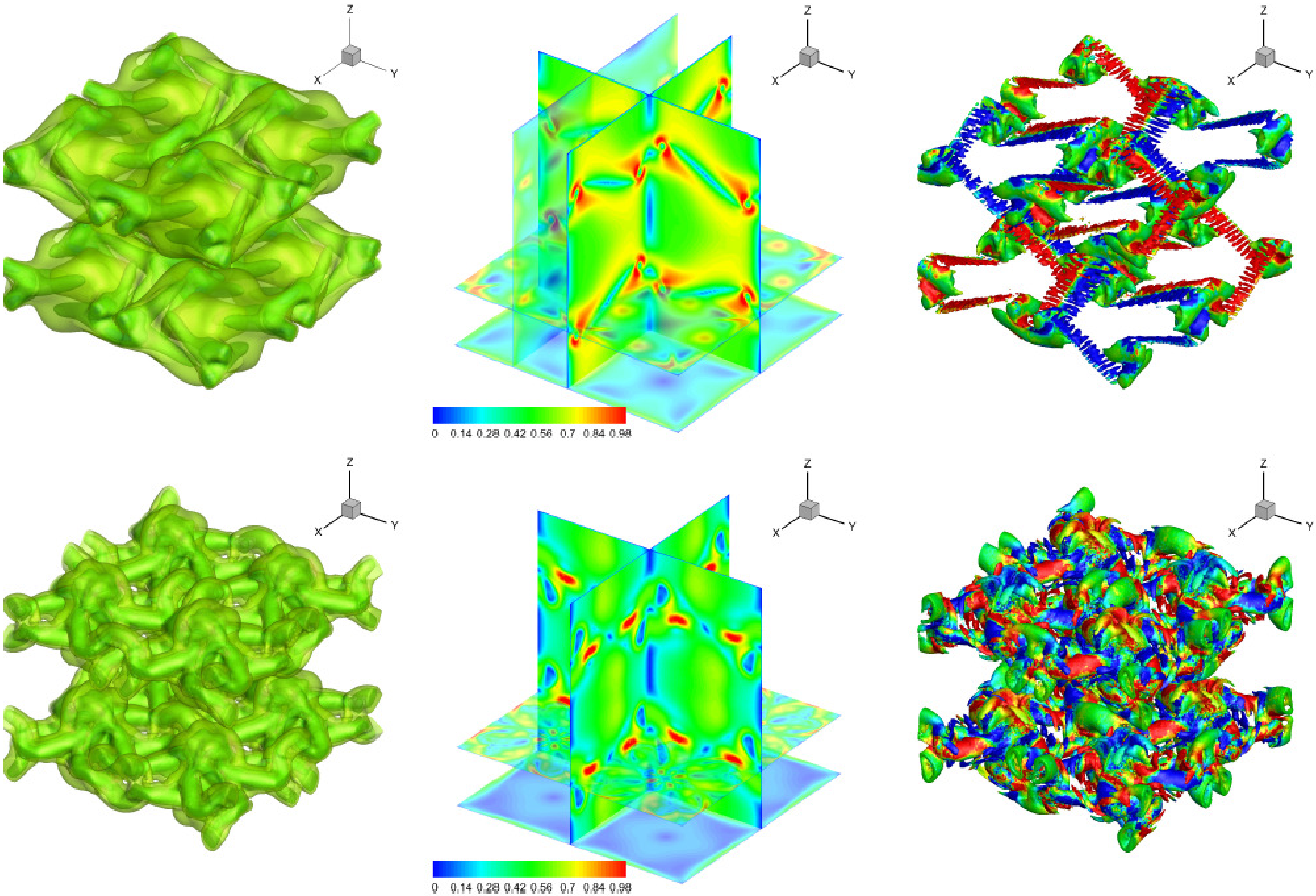
3D Arnold-Beltrami-Childress Flow

p	p_γ	N_e	$\epsilon(p)$	$\epsilon(\mathbf{v})$	$\sigma(p)$	$\sigma(\mathbf{v})$
1	1	10368	1.1713E+00	2.4695E-01	1.6	2.0
1	1	13182	1.0388E+00	2.1017E-01	1.5	2.0
1	1	16464	9.2718E-01	1.8075E-01	1.5	2.0
1	1	20250	8.3860E-01	1.5730E-01	1.5	2.0
2	2	10368	1.7339E-01	1.4475E-02	2.8	3.1
2	2	13182	1.4060E-01	1.1291E-02	2.6	3.1
2	2	16464	1.1470E-01	8.9676E-03	2.8	3.1
2	2	20250	9.5780E-02	7.2516E-03	2.6	3.1
3	3	6000	1.6219E-02	1.5469E-03	3.8	4.1
3	3	7986	1.1454E-02	1.0494E-03	3.7	4.1
3	3	10368	8.2191E-03	7.3591E-04	3.8	4.1
3	3	13182	6.1399E-03	5.3142E-04	3.6	4.1
4	4	750	4.5578E-02	3.2574E-03	4.7	4.8
4	4	1296	1.9664E-02	1.2957E-03	4.6	5.1
4	4	2058	9.3757E-03	5.9049E-04	4.8	5.1
4	4	3072	5.0553E-03	2.9738E-04	4.6	5.1

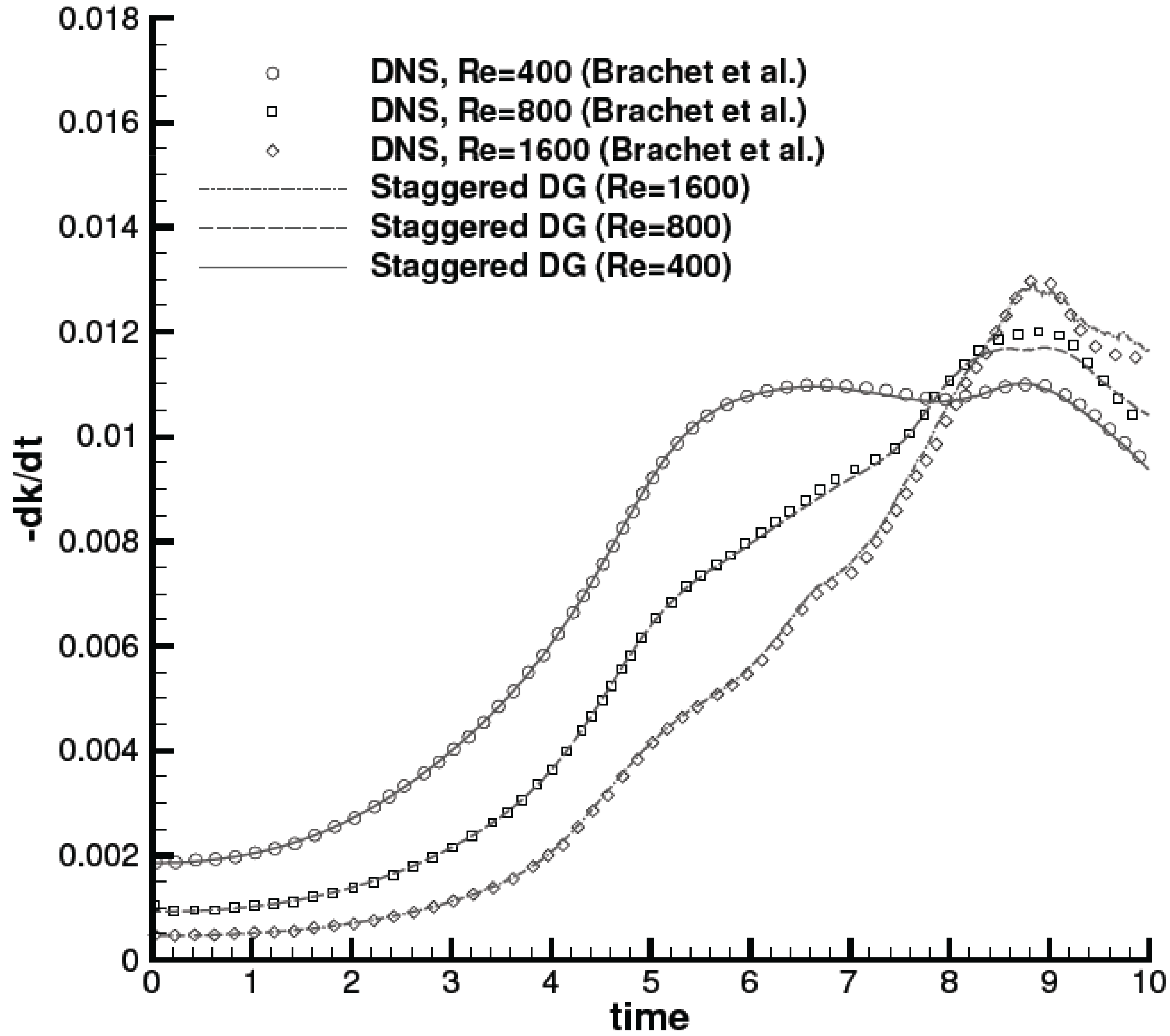
3D Taylor-Green Vortex



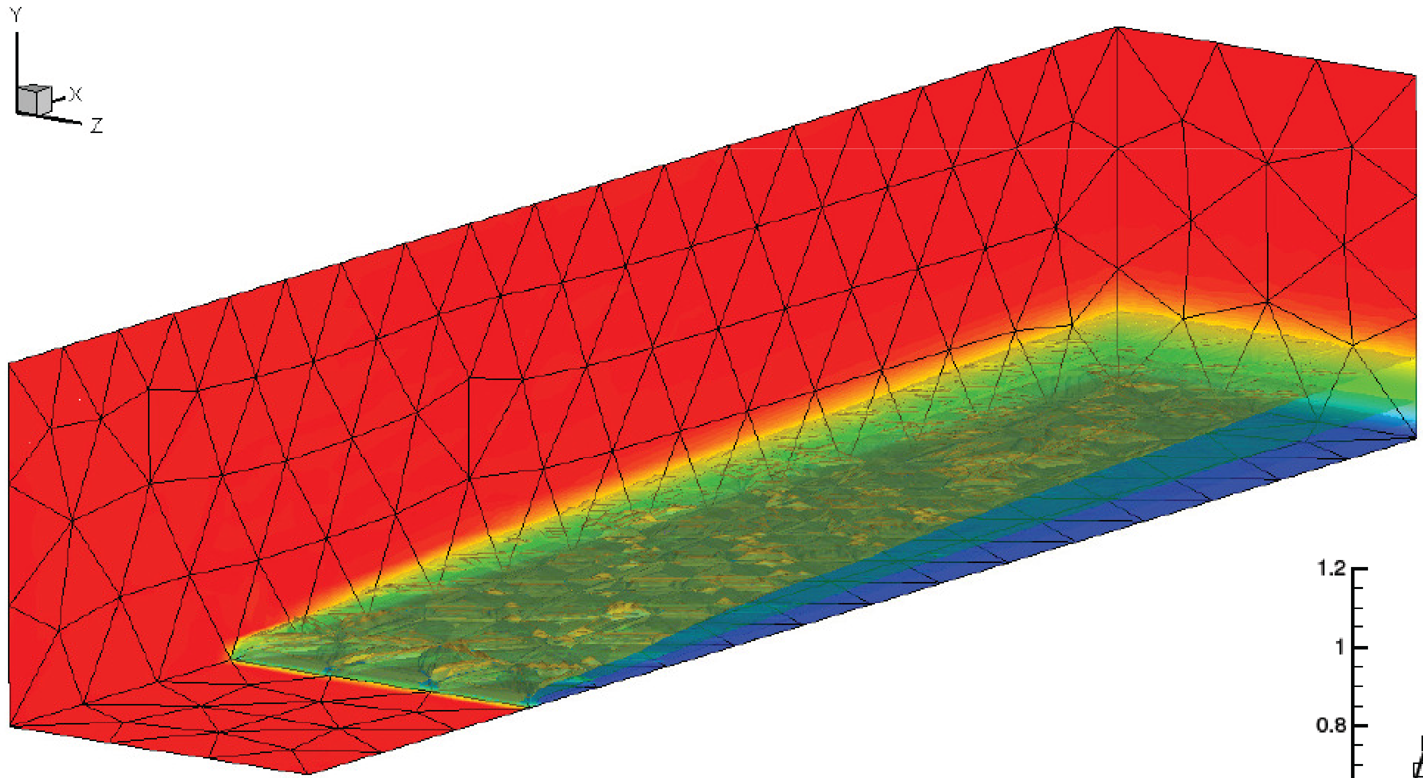
3D Taylor-Green Vortex

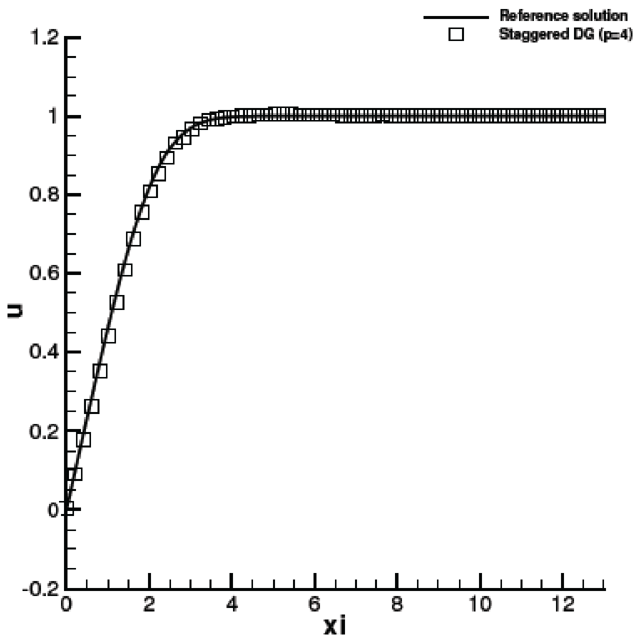


3D Taylor-Green Vortex

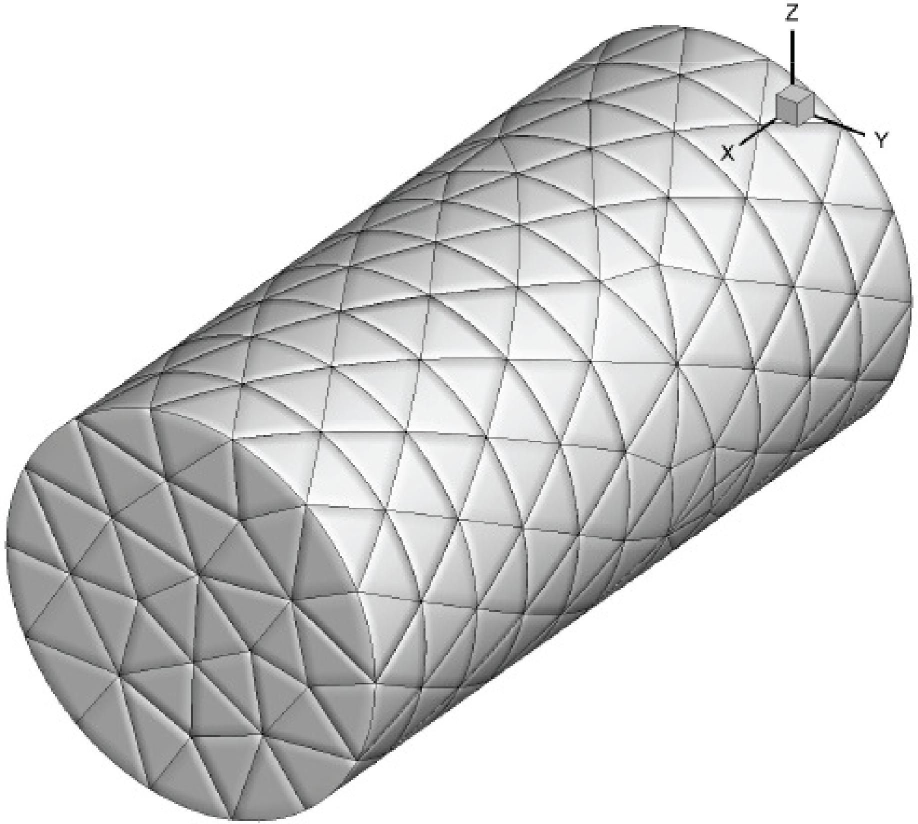


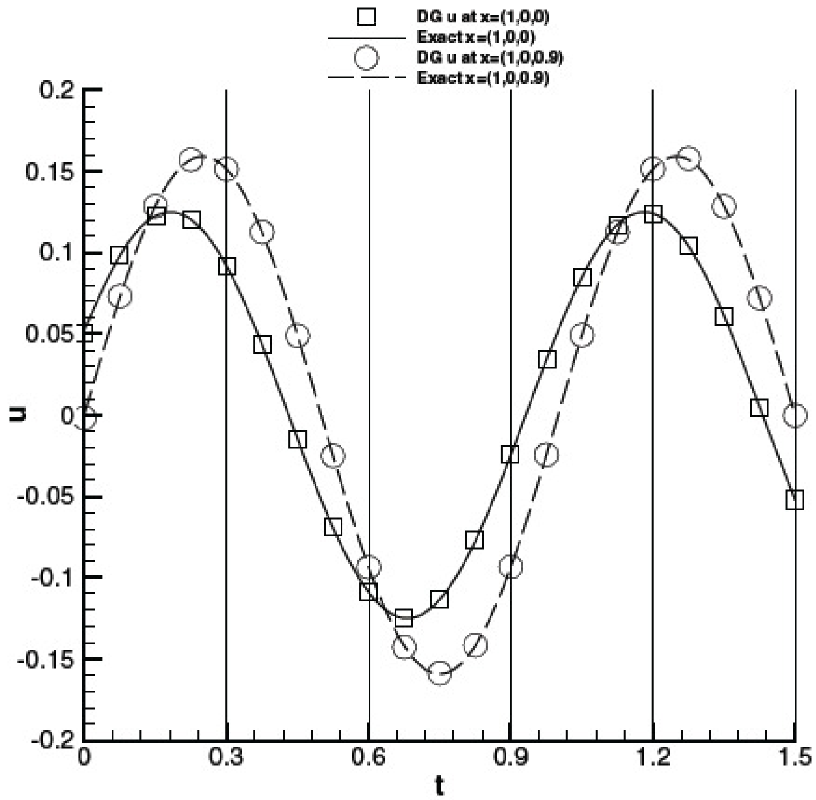
3D Laminar Boundary Layer





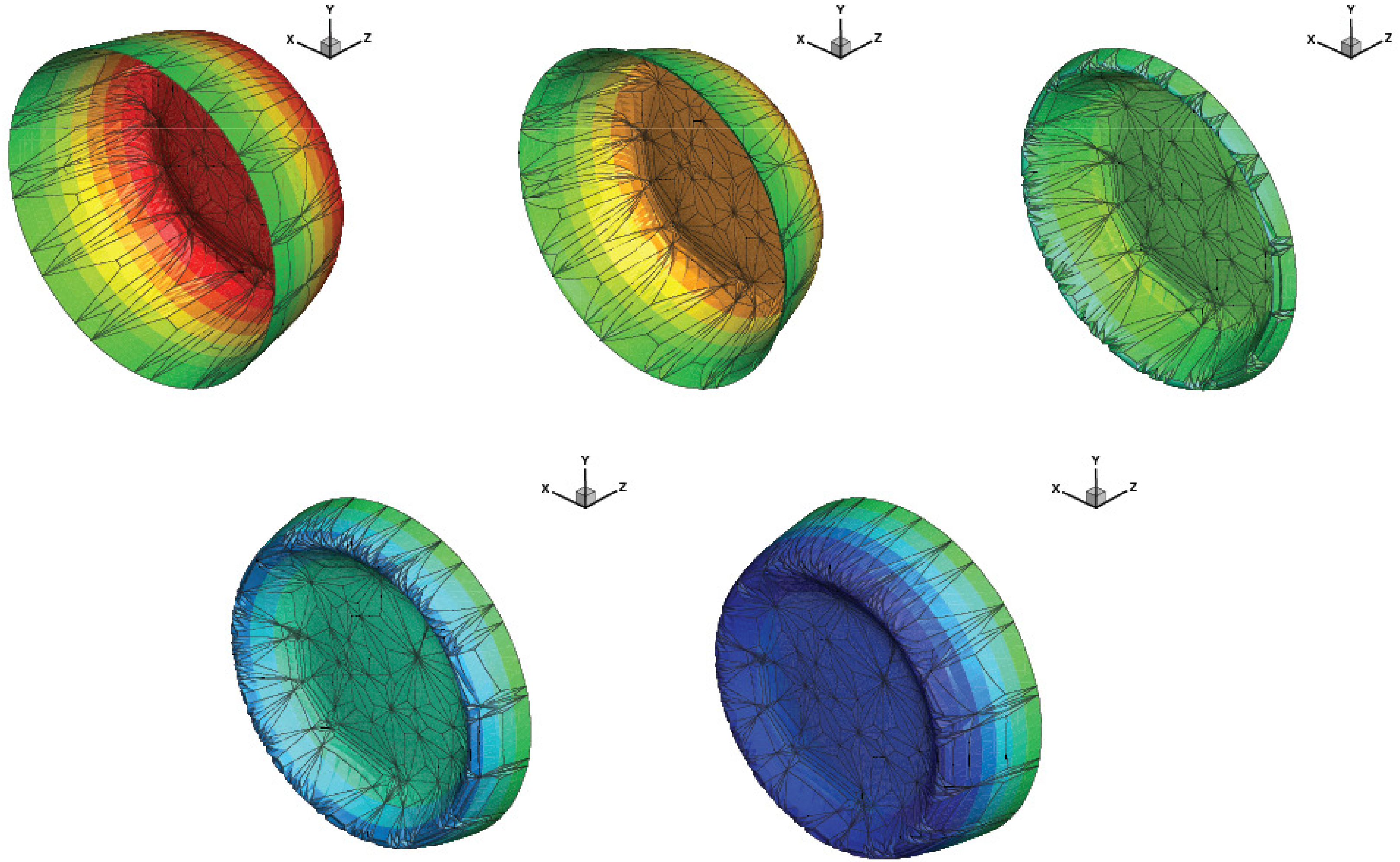
3D Womersley Flow





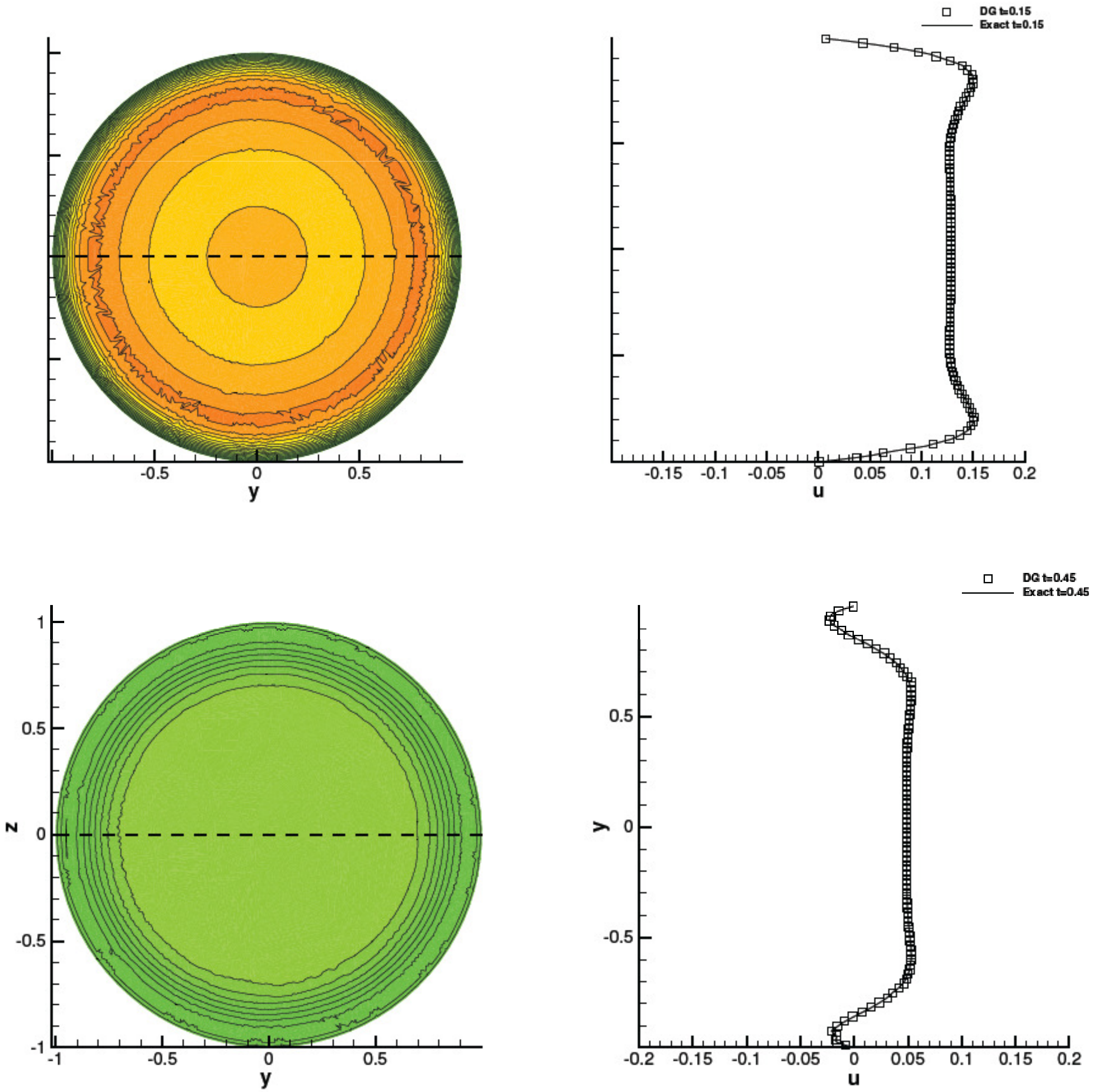
Here, we use high order **curved** isoparametric elements.

3D Womersley Flow

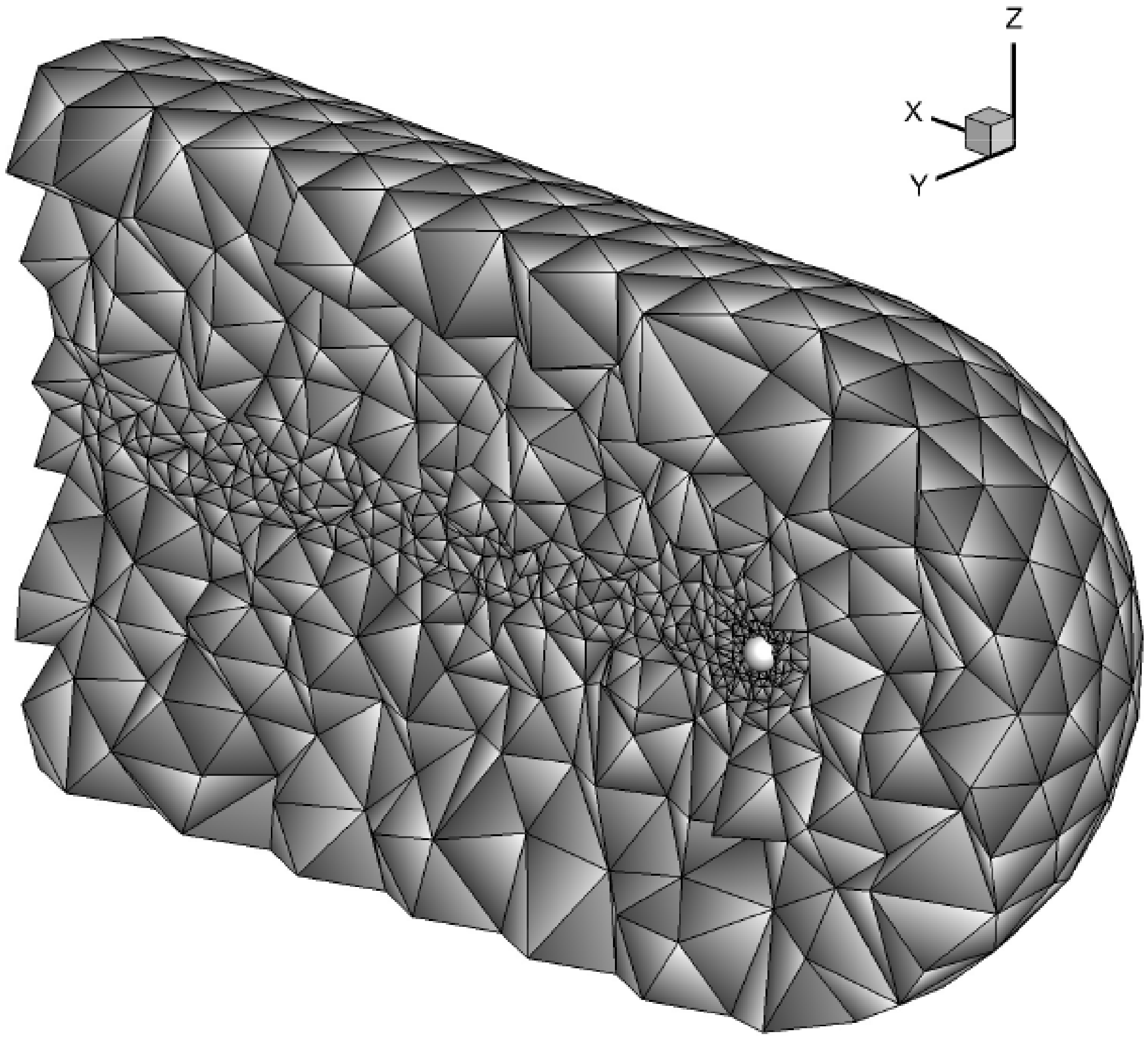


The oscillating profile in a slice within the y-z plane during **one single timestep!**
Evaluation of the space-time polynomials is possible at any intermediate time.

3D Womersley Flow

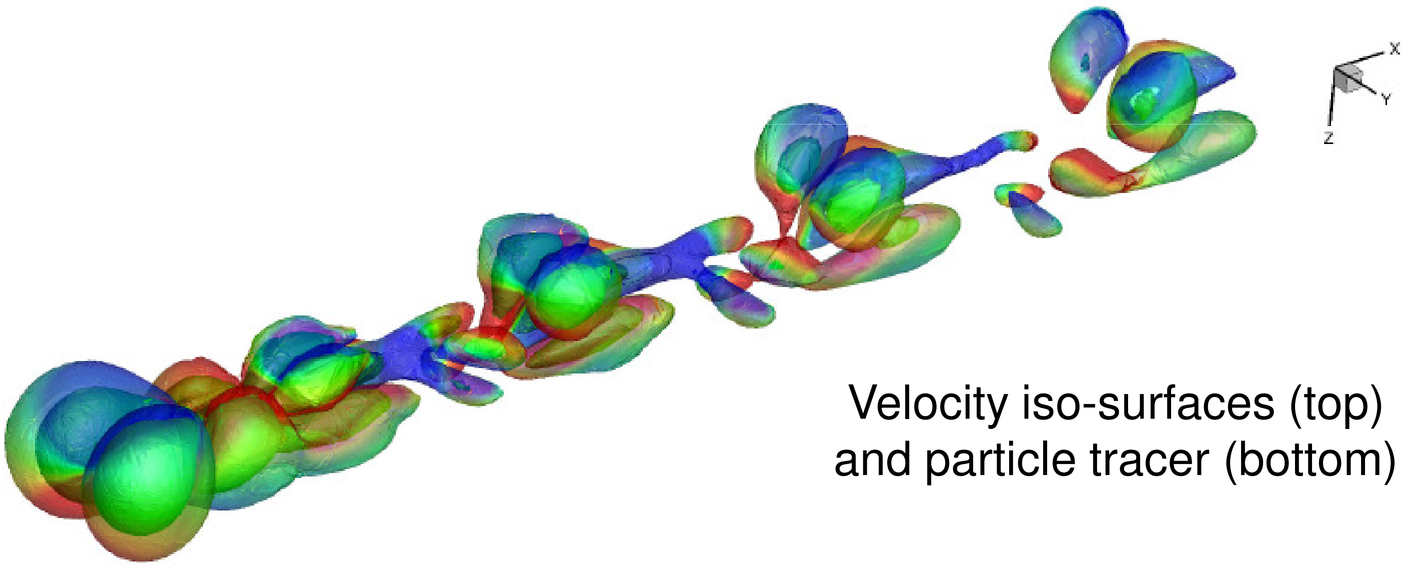


3D flow around a sphere

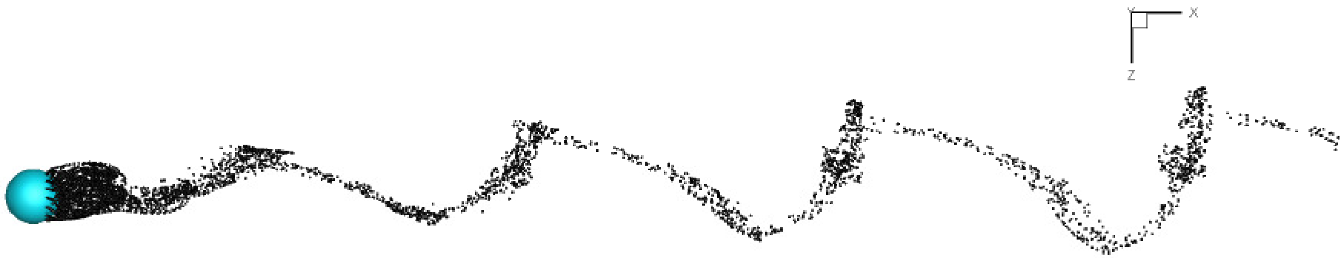


Here, we use high order **curved** isoparametric elements.

3D flow around a sphere



Velocity iso-surfaces (top) and particle tracer (bottom)



Conclusions & Outlook

- The first extension of the semi-implicit UnTRIM method of Casulli et al. to **arbitrary** high order of accuracy in **space** and **time** for the shallow water and the incompressible Navier-Stokes equations.

- Large time steps, which is particularly important for DG FEM schemes that have an explicit CFL stability restriction of CFL < 1/(2N+1) !!

- The very high order scheme is much more accurate compared to the N=0 case due to high order accurate subgrid resolution of the **full hydrodynamics**

- Future extensions to 3D hydrostatic and non-hydrostatic flows; applications to blood flow and compressible flows in compliant vessels; design of new a posteriori subcell limiters for staggered semi-implicit DG schemes (M. Ioriatti)

- Implementation of a spectral staggered DG scheme on space-time adaptive (AMR) Cartesian grids (F. Fambri)

- Extension to the full **compressible Navier-Stokes** equations (M. Tavelli)

- Extension to the viscous and resistive **MHD equations**

List of References

[1] M. Dumbser and V. Casulli. **A Staggered Semi-Implicit Spectral Discontinuous Galerkin Scheme for the Shallow Water Equations**. *Applied Mathematics and Computation*, **219**:8057-8077, 2013

[2] M. Tavelli and M. Dumbser. **A high order semi-implicit discontinuous Galerkin method for the two dimensional shallow water equations on staggered unstructured meshes**. *Applied Mathematics and Computation*, **234**:623–644, 2014.

[3] M. Tavelli and M. Dumbser. **A staggered semi-implicit discontinuous Galerkin method for the two dimensional incompressible Navier-Stokes equations**, *Applied Mathematics and Computation*, **248**:70–92, 2014

[4] M. Tavelli and M. Dumbser. **A staggered space-time discontinuous Galerkin method for the incompressible Navier-Stokes equations on two-dimensional triangular meshes.** *Computers and Fluids*, **119**:235-249, 2015

[5] M. Tavelli and M. Dumbser. **A staggered space-time discontinuous Galerkin method for the three-dimensional incompressible Navier-Stokes equations on unstructured tetrahedral meshes**. *Journal of Computational Physics*. **319**:294–323, 2016.