



# *A simple robust and accurate a posteriori subcell finite volume limiter for the discontinuous Galerkin method*

Michael Dumbser

ERC Grant *STiMuUs*

FET-HPC Grant *ExaHyPE*



## Objectives

- (1) Design a new limiter for the discontinuous Galerkin finite element method that is **simple, robust and accurate**
- (2a) The new limiter **must not destroy** the subcell resolution capability of the DG scheme, neither at discontinuities, nor in smooth regions, where it might have been erroneously activated, or, equivalently
- (2b) The limiter must act on a characteristic **length scale** of  $h/(N+1)$  and **not** on the length scale  $h$  of the main grid, i.e. accuracy improves with  $N$  **even at shocks**
- (3) The DG limiter should **not** contain **problem-dependent parameters**, like, e.g., the well-known parameter  $M$  of the classical TVB limiter of Cockburn and Shu.
- (4) The new limiter should work well for **very high** polynomial degrees, say  $N=9$ .
- (5) Ideally, the final DG scheme should become **as robust** as a traditional **second order TVD finite volume scheme**, but **more accurate** on a given computational mesh of characteristic mesh size  $h$

# Unlimited Fully Discrete One-Step ADER-DG Scheme

Governing hyperbolic PDE system of the form

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{Q}) = 0 \quad (\text{PDE})$$

with the vector of conserved variables  $\mathbf{Q}$  and the nonlinear flux tensor  $\mathbf{F}(\mathbf{Q})$ .  
The discrete solution at time  $t^n$  is represented by piecewise polynomials of degree  $N$  over spatial control volumes  $T_i$  as

$$\mathbf{u}_h(\mathbf{x}, t^n) = \sum_l \Phi_l(\mathbf{x}) \hat{\mathbf{u}}_l^n, \quad \mathbf{x} \in T_i \quad (\text{DG})$$

Multiplication with a test function  $\phi_k$  from the space of piecewise polynomials of degree  $N$  and integration over a space-time control volume  $T_i \times [t^n, t^{n+1}]$  yields:

$$\int_{t^n}^{t^{n+1}} \int_{T_i} \Phi_k \frac{\partial \mathbf{Q}}{\partial t} d\mathbf{x} dt + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \mathbf{F}(\mathbf{Q}) \cdot \mathbf{n} dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}(\mathbf{Q}) d\mathbf{x} dt = 0$$

## Unlimited Fully Discrete One-Step ADER-DG Scheme

We then introduce the discrete solution (DG) and an **element-local space-time predictor**  $\mathbf{q}_h(\mathbf{x}, t)$ , together with a classical (monotone) numerical flux  $G$ , as it is used in Godunov-type finite volume schemes.

The fully discrete one-step ADER-DG scheme then simply reads:

$$\left( \int_{T_i} \Phi_k \Phi_l d\mathbf{x} \right) (\hat{\mathbf{u}}_l^{n+1} - \hat{\mathbf{u}}_l^n) + \int_{t^n}^{t^{n+1}} \int_{\partial T_i} \Phi_k \mathcal{G}(\mathbf{q}_h^-, \mathbf{q}_h^+) \cdot \mathbf{n} dS dt - \int_{t^n}^{t^{n+1}} \int_{T_i} \nabla \Phi_k \cdot \mathbf{F}(\mathbf{q}_h) d\mathbf{x} dt = 0$$

But how to compute the space-time predictor  $\mathbf{q}_h(\mathbf{x}, t)$ , since at the beginning of a time step, only the discrete spatial solution  $\mathbf{u}_h(\mathbf{x}, t^n)$  at time  $t^n$  is known?

Use a weak integral form of the PDE in space-time and solve an element-local Cauchy problem *in the small*, with initial data  $\mathbf{u}_h(\mathbf{x}, t^n)$ , similar to the MUSCL-Hancock scheme or the ENO scheme of Harten et al.

## Element-local Space-time Predictor

Rewrite the governing PDE in a reference coordinate system  $\xi$ - $\tau$  on a reference element  $T_E$ :

$$\frac{\partial \mathbf{Q}}{\partial \tau} + \nabla_{\xi} \cdot \mathbf{F}^*(\mathbf{Q}) = 0, \quad \mathbf{F}^* := \Delta t (\partial \xi / \partial \mathbf{x})^T \cdot \mathbf{F}(\mathbf{Q}).$$

We introduce the two space-time integral operators

$$\langle f, g \rangle = \int_0^1 \int_{T_E} (f(\xi, \tau) \cdot g(\xi, \tau)) d\xi d\tau, \quad [f, g]^{\tau} = \int_{T_E} (f(\xi, \tau) \cdot g(\xi, \tau)) d\xi.$$

The discrete space-time predictor solution and the discrete flux are defined as

$$\mathbf{q}_h = \mathbf{q}_h(\xi, \tau) = \sum_l \theta_l(\xi, \tau) \hat{\mathbf{q}}_l := \theta_l \hat{\mathbf{q}}_l, \quad \text{nodal space-time basis } \theta_l$$

$$\mathbf{F}_h^* = \mathbf{F}_h^*(\xi, \tau) = \sum_l \theta_l(\xi, \tau) \hat{\mathbf{F}}_l^* := \theta_l \hat{\mathbf{F}}_l^*, \quad \hat{\mathbf{F}}_l^* = \mathbf{F}^*(\hat{\mathbf{q}}_l).$$

## Element-local Space-time Predictor

Multiplication with a **space-time** test function and integration over the space-time reference element  $T_E \times [0,1]$  yields:

$$\left\langle \theta_k, \frac{\partial \mathbf{q}_h}{\partial \tau} \right\rangle + \langle \theta_k, \nabla_\xi \cdot \mathbf{F}_h^*(\mathbf{q}_h) \rangle = 0.$$

The initial condition  $\mathbf{u}_h(\mathbf{x}, t^n)$  is introduced in a **weak sense** after integration by parts **in time** (upwinding in time, causality principle):

$$[\theta_k, \mathbf{q}_h]^1 - [\theta_k, \mathbf{u}_h]^0 - \left\langle \frac{\partial}{\partial \tau} \theta_k, \mathbf{q}_h \right\rangle + \langle \theta_k, \nabla_\xi \cdot \mathbf{F}_h^* \rangle = 0.$$

The above element-local nonlinear system is easily solved via the following fast-converging fixed-point iteration (discrete Picard iteration):

$$\left( [\theta_k, \theta_l]^1 - \left\langle \frac{\partial}{\partial \tau} \theta_k, \theta_l \right\rangle \right) \hat{\mathbf{q}}_l^{r+1} = [\theta_k, \Phi_l]^0 \hat{\mathbf{u}}_l^n - \langle \theta_k, \nabla_\xi \theta_l \rangle \cdot \mathbf{F}^*(\hat{\mathbf{q}}_l^r),$$

## A new *a posteriori* limiter of DG-FEM methods

- Motivation: develop a **simple**, **robust** and **parameter-free** limiter for DG that **always works** and which does **not destroy** the **subcell resolution** of DG
- Conventional DG limiters use either artificial viscosity, which needs parameters to be tuned, or nonlinear FV-type reconstruction/limiters (TVB, WENO, HWENO), which **usually destroy** the subcell resolution properties.
- Our new approach: extend the successful *a posteriori* **MOOD** method of Loubère et al., developed in the FV context, also to the DG-FEM framework.
- As very simple *a posteriori* detection criteria, we only use
  - A relaxed discrete maximum principle (**DMP**) in the sense of polynomials
  - **Positivity** of the solution and absence of floating point errors (**NaN**)
- If one of these criteria is violated after a time step, the scheme **goes back** to the old time step and **recomputes** the solution in the troubled cells, using a more robust ADER-WENO or TVD FV scheme **on a fine subgrid** composed of  **$2N+1$**  subcells per space dimension

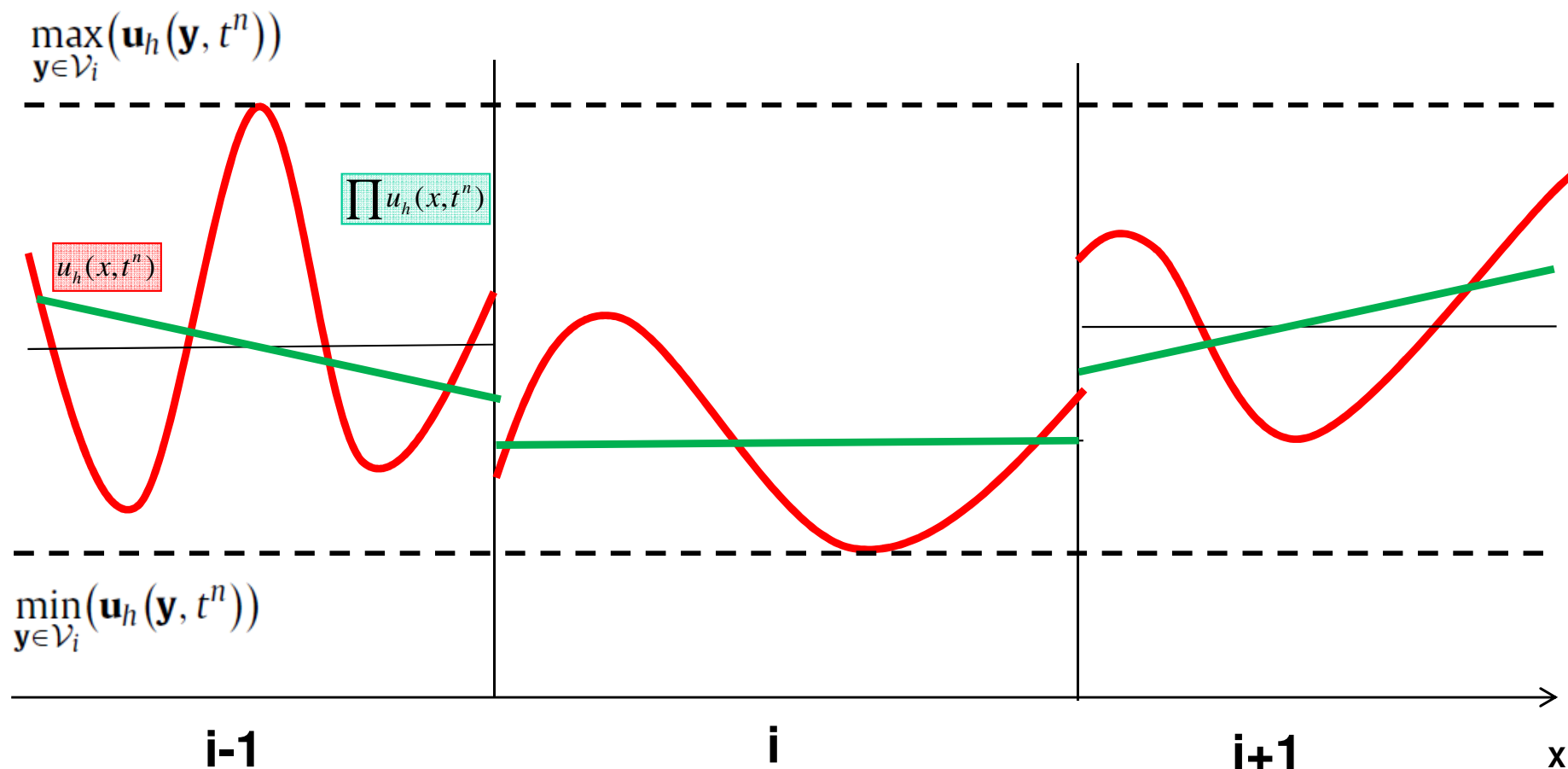


## *A new a posteriori limiter of DG-FEM methods*

- Classical DG limiters, like WENO/HWENO/slope/moment limiters are based on **nonlinear data post-processing**, while the new DG limiter **recomputes** the discrete solution with a more robust scheme, starting again from a **valid solution** available at the old time level
- Alternative description: dynamic, element-local **checkpointing** and **restarting** of the solver with a more robust scheme on a finer grid
- This enables the limiter even to **cure** floating point errors (**NaN** values appearing after division by zero or after taking roots of negative numbers)
- The new method is by construction **positivity preserving**, if the underlying finite volume scheme on the subgrid preserves positivity
- **Local limiter** (in contrast to WENO limiters for DG), since it requires only information from the cell and its direct neighborhood
- As **accurate** as a high order **unlimited DG scheme** in smooth flow regions, but at the same time as **robust** as a **second order TVD scheme** at shocks or other discontinuities, but also at strong rarefactions

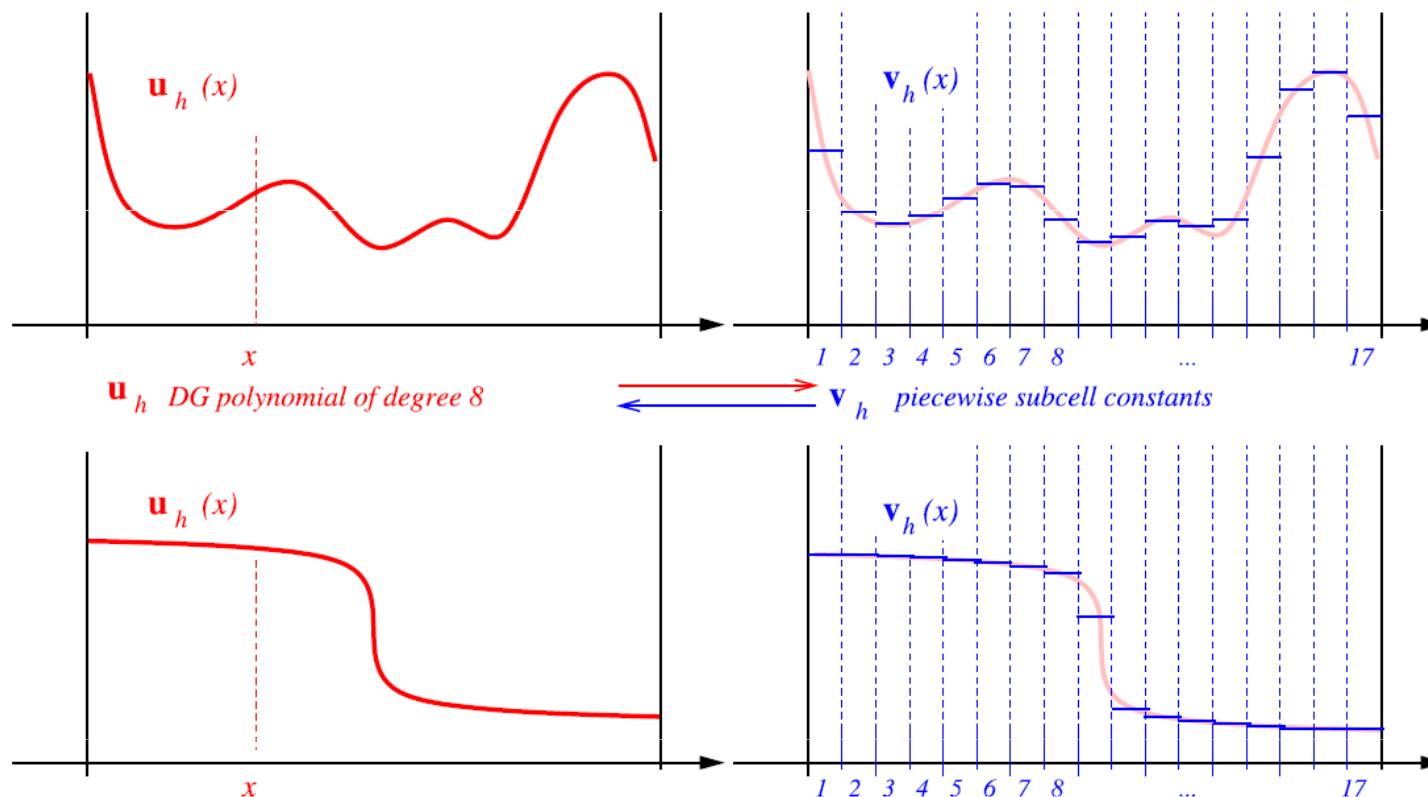


## Classical TVB slope/moment limiting of DG



If a classical nonlinear reconstruction-based DG limiter is activated erroneously, there may be important physical information that is lost forever!

## A new *a posteriori* limiter of DG-FEM methods



DG polynomials of degree  $N=8$  (left) and **equivalent** data representation on  $2N+1=17$  **subcells** (right). Arrows indicate projection (red) and reconstruction (blue)

$$\mathcal{R} \circ \mathcal{P} = \mathcal{I}$$

We use  $2N+1$  subcells to **match** the DG time step ( $\text{CFL} < 1/(2N+1)$ ) on the coarse grid with the FV time step ( $\text{CFL} < 1$ ) on the fine subgrid.

## A new *a posteriori* limiter of DG-FEM methods

Projection from the DG polynomials to the subcell averages

$$\mathbf{v}_{i,j}^n = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \frac{1}{|S_{i,j}|} \int_{S_{i,j}} \phi_l(\mathbf{x}) d\mathbf{x} \hat{\mathbf{u}}_l^n, \quad \forall S_{i,j} \in \mathcal{S}_i.$$

Reconstruction of DG polynomials from the subcell averages

$$\int_{S_{i,j}} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{S_{i,j}} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}, \quad \forall S_{i,j} \in \mathcal{S}_i.$$

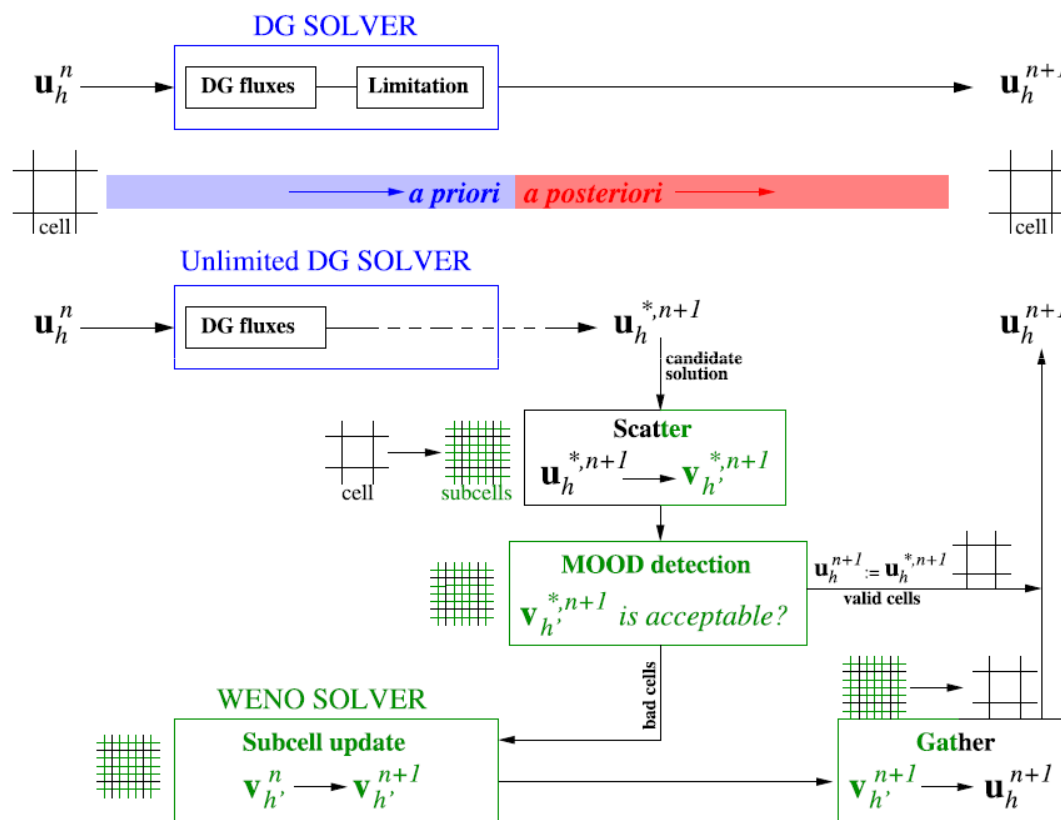
$$\int_{T_i} \mathbf{u}_h(\mathbf{x}, t^n) d\mathbf{x} = \int_{T_i} \mathbf{v}_h(\mathbf{x}, t^n) d\mathbf{x}. \quad \text{Linear constraint: conservation}$$

# *A posteriori* detection criteria and DG-MOOD flowchart

**Positivity:**  $\pi_k(\mathbf{u}_h^*(\mathbf{x}, t^{n+1})) > 0,$

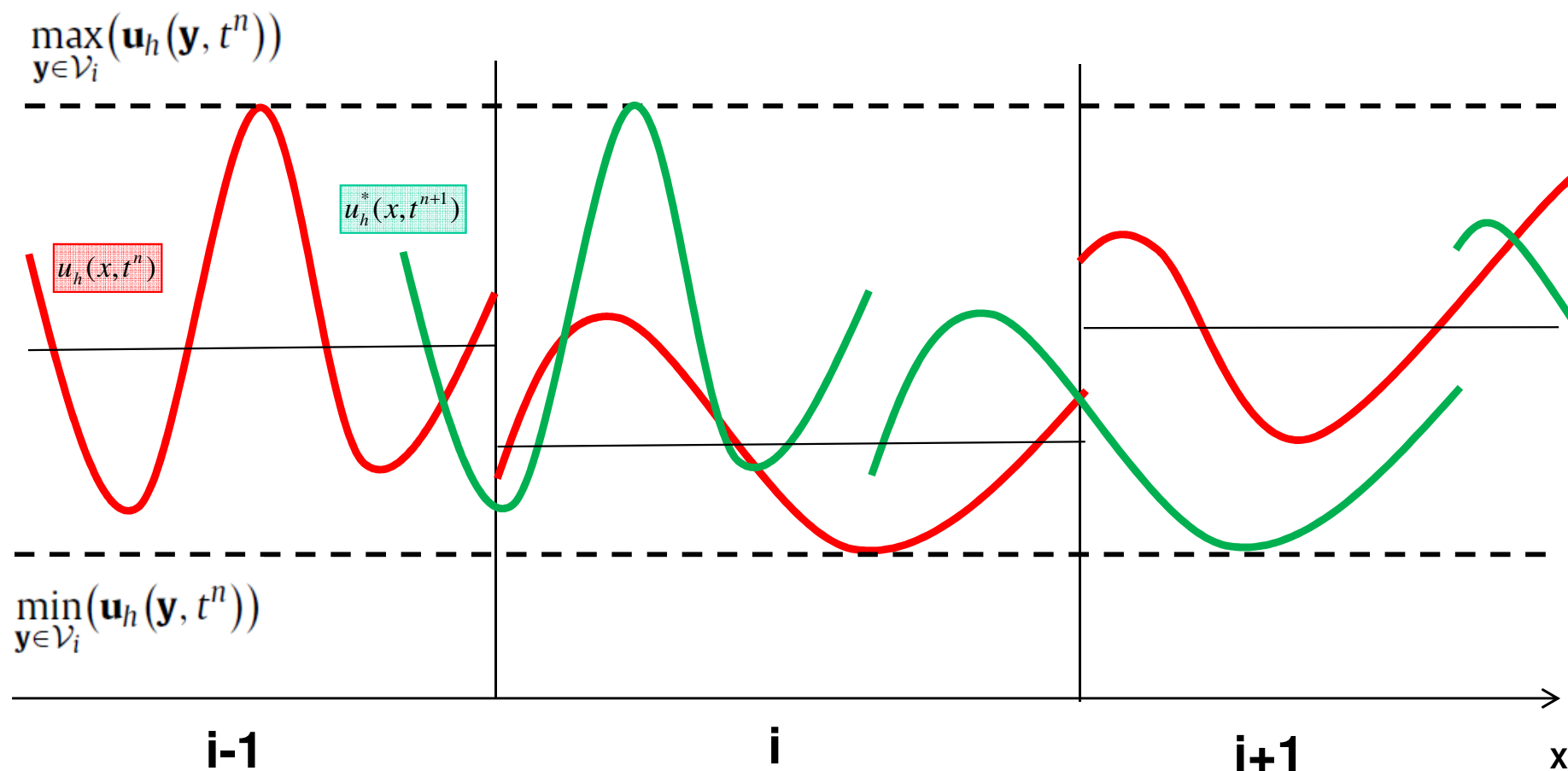
**Relaxed DMP in the sense of polynomials:**

$$\min_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{u}_h(\mathbf{y}, t^n)) - \delta \leq \mathbf{u}_h^*(\mathbf{x}, t^{n+1}) \leq \max_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{u}_h(\mathbf{y}, t^n)) + \delta,$$



## DMP in the sense of polynomials

$$\min_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{u}_h(\mathbf{y}, t^n)) - \delta \leq \mathbf{u}_h^*(\mathbf{x}, t^{n+1}) \leq \max_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{u}_h(\mathbf{y}, t^n)) + \delta,$$



## Summary of the ADER-DG-MOOD scheme

Verification of the DMP and the positivity on the **candidate solution**  $\mathbf{u}_h^*(\mathbf{x}, t^{n+1})$ :

$$\min_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{v}_h(\mathbf{y}, t^n)) - \delta \leq \mathbf{v}_h^*(\mathbf{x}, t^{n+1}) \leq \max_{\mathbf{y} \in \mathcal{V}_i}(\mathbf{v}_h(\mathbf{y}, t^n)) + \delta, \quad \forall \mathbf{x} \in T_i,$$

$$\pi_k(\mathbf{u}_h^*(\mathbf{x}, t^{n+1})) > 0, \quad \forall \mathbf{x} \in T_i, \quad \forall k,$$

If a cell does not satisfy both criteria, flag it as troubled cell,  $\beta_i^{n+1} = 1$ , **discard** the DG solution and **recompute** it with a more robust third order **ADER-WENO** or an even more robust **second order TVD finite** volume scheme on the **fine subgrid**:

$$\mathbf{v}_h(\mathbf{x}, t^{n+1}) = \mathcal{A}(\mathbf{v}_h(\mathbf{x}, t^n))$$

$$\mathbf{v}_h(\mathbf{x}, t^n) = \begin{cases} \mathcal{P}(\mathbf{u}_h(\mathbf{x}, t^n)) & \text{if } \beta_j^n = 0, \\ \mathcal{A}(\mathbf{v}_h(\mathbf{x}, t^{n-1})) & \text{if } \beta_j^n = 1. \end{cases} \quad \mathbf{x} \in T_j \quad \forall T_j \in \mathcal{V}_i.$$

Finally, **reconstruct** the DG polynomial from the subcell averages:

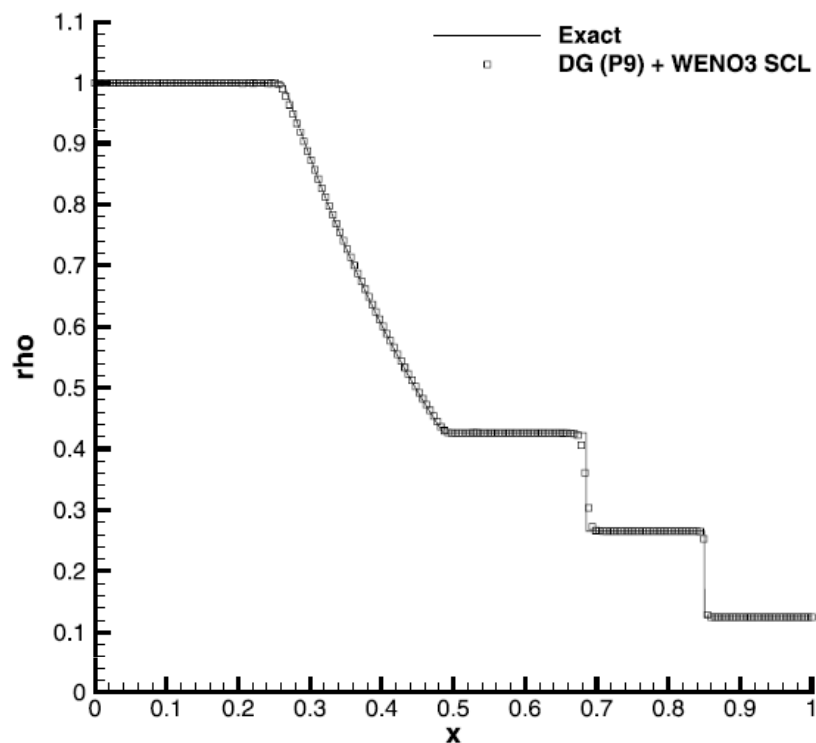
$$\mathbf{u}_h(\mathbf{x}, t^{n+1}) = \mathcal{R}(\mathbf{v}_h(\mathbf{x}, t^{n+1})) \quad \text{or} \quad \mathbf{u}_h(\mathbf{x}, t^{n+1}) = \mathcal{R}(\mathcal{A}(\mathbf{v}_h(\mathbf{x}, t^n)))$$

## 2D Numerical Convergence Results P2-P9 (Euler)

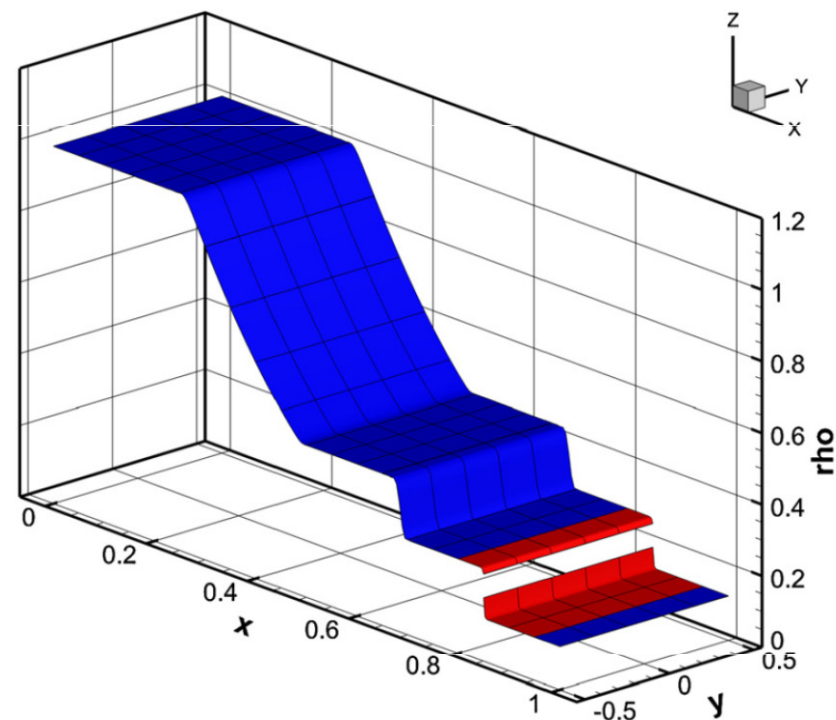
	$N_x$	$L^1$ error	$L^2$ error	$L^\infty$ error	$L^1$ order	$L^2$ order	$L^\infty$ order	Theor.
DG-P <sub>2</sub>	25	9.33E-03	2.07E-03	2.02E-03	–	–	–	3
	50	6.70E-04	1.58E-04	1.66E-04	3.80	3.71	3.60	
	75	1.67E-04	4.07E-05	4.45E-05	3.43	3.35	3.25	
	100	6.74E-05	1.64E-05	1.82E-05	3.15	3.15	3.10	
DG-P <sub>3</sub>	25	5.77E-04	9.42E-05	7.84E-05	–	–	–	4
	50	2.75E-05	4.52E-06	4.09E-06	4.39	4.38	4.26	
	75	4.36E-06	7.89E-07	7.55E-07	4.55	4.30	4.17	
	100	1.21E-06	2.37E-07	2.38E-07	4.46	4.17	4.01	
DG-P <sub>4</sub>	20	1.54E-04	2.18E-05	2.20E-05	–	–	–	5
	30	1.79E-05	2.46E-06	2.13E-06	5.32	5.37	5.75	
	40	3.79E-06	5.35E-07	5.18E-07	5.39	5.31	4.92	
	50	1.11E-06	1.61E-07	1.46E-07	5.50	5.39	5.69	
DG-P <sub>5</sub>	10	9.72E-04	1.59E-04	2.00E-04	–	–	–	6
	20	1.56E-05	2.13E-06	2.14E-06	5.96	6.22	6.55	
	30	1.14E-06	1.64E-07	1.91E-07	6.45	6.33	5.96	
	40	2.17E-07	2.97E-08	3.59E-08	5.77	5.93	5.82	
DG-P <sub>6</sub>	5	2.24E-02	4.15E-03	3.11E-03	–	–	–	7
	10	1.76E-04	2.75E-05	2.86E-05	6.99	7.24	6.76	
	20	1.67E-06	2.28E-07	2.26E-07	6.72	6.91	6.98	
	25	3.60E-07	4.96E-08	6.27E-08	6.86	6.84	5.74	
DG-P <sub>7</sub>	5	5.50E-03	1.22E-03	1.46E-03	–	–	–	8
	10	4.63E-05	6.26E-06	6.95E-06	6.89	7.61	7.71	
	15	1.62E-06	2.20E-07	2.29E-07	8.28	8.26	8.42	
	20	2.05E-07	2.80E-08	2.28E-08	7.18	7.17	8.01	
DG-P <sub>8</sub>	4	9.11E-03	1.80E-03	3.44E-03	–	–	–	9
	8	4.97E-05	7.51E-06	6.93E-06	7.52	7.90	8.96	
	10	7.50E-06	1.05E-06	1.18E-06	8.47	8.81	7.95	
	15	2.40E-07	3.34E-08	3.09E-08	8.49	8.51	8.98	
DG-P <sub>9</sub>	4	3.95E-03	7.89E-04	1.42E-03	–	–	–	10
	8	1.01E-05	1.44E-06	1.52E-06	8.61	9.09	9.87	
	10	1.44E-06	2.00E-07	2.27E-07	8.74	8.85	8.51	
	12	2.67E-07	3.70E-08	3.77E-08	9.26	9.25	9.85	



## ADER-DG-MOOD Results

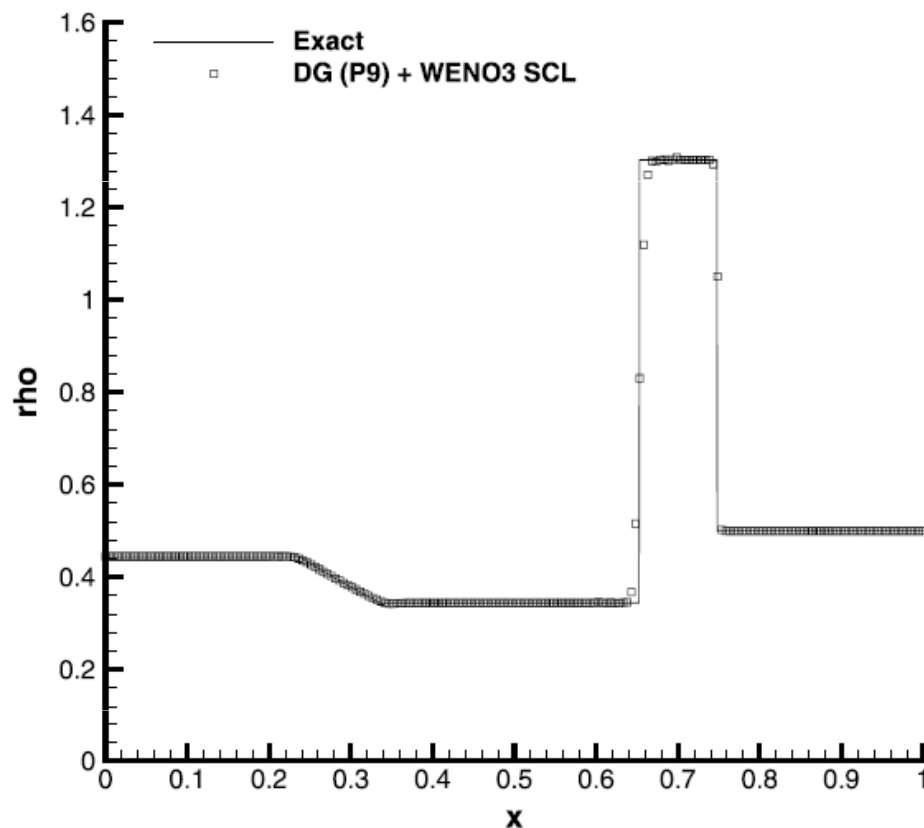


Sod shock tube,  
20x5 elements (**N=9**)

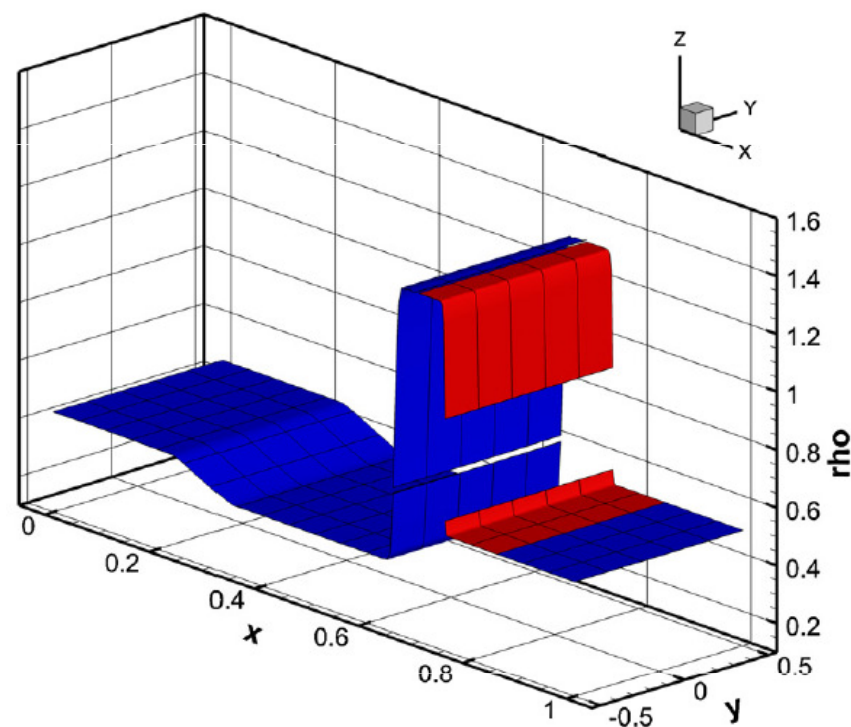


Limited cells (red),  
Unlimited cells (blue)

## ADER-DG-MOOD Results

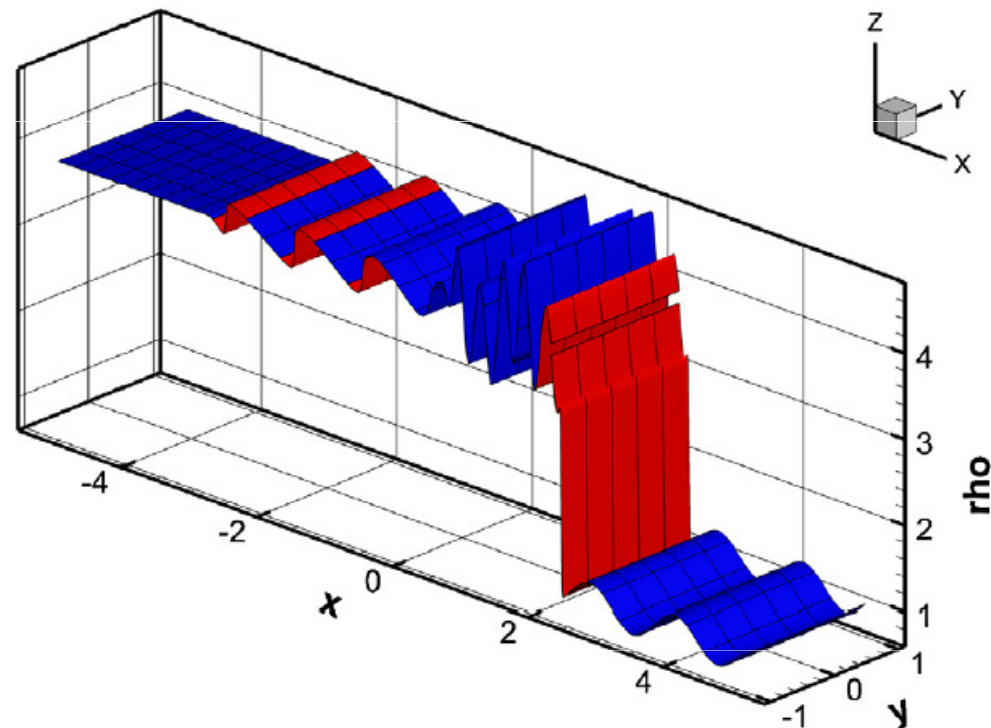
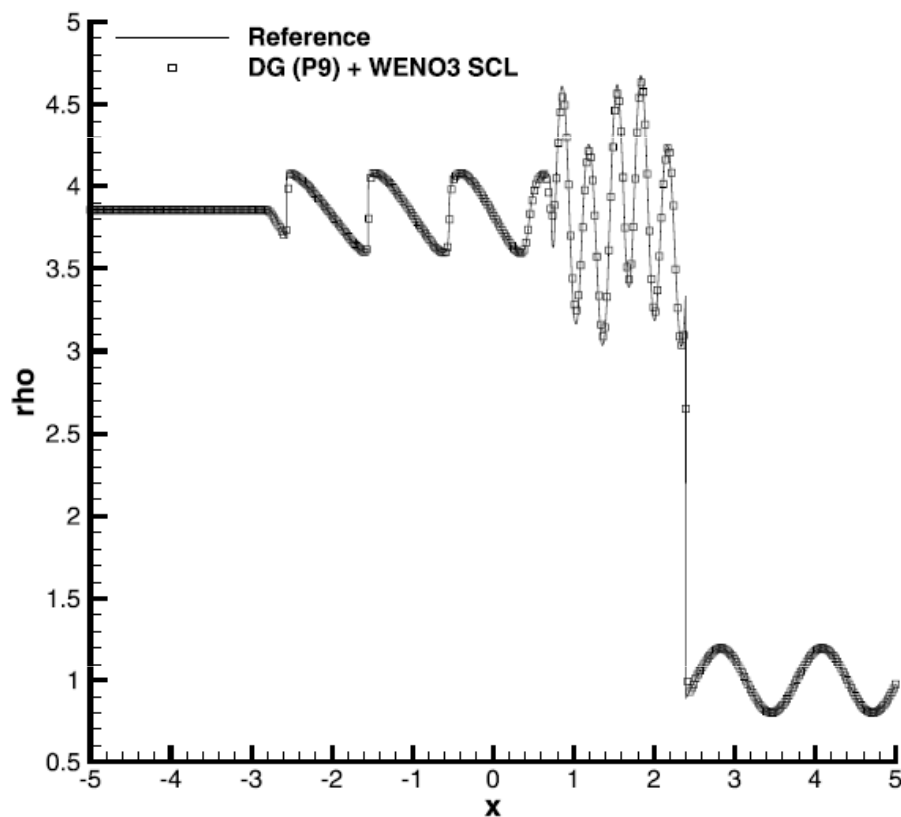


Lax shock tube,  
20x5 elements (**N=9**)



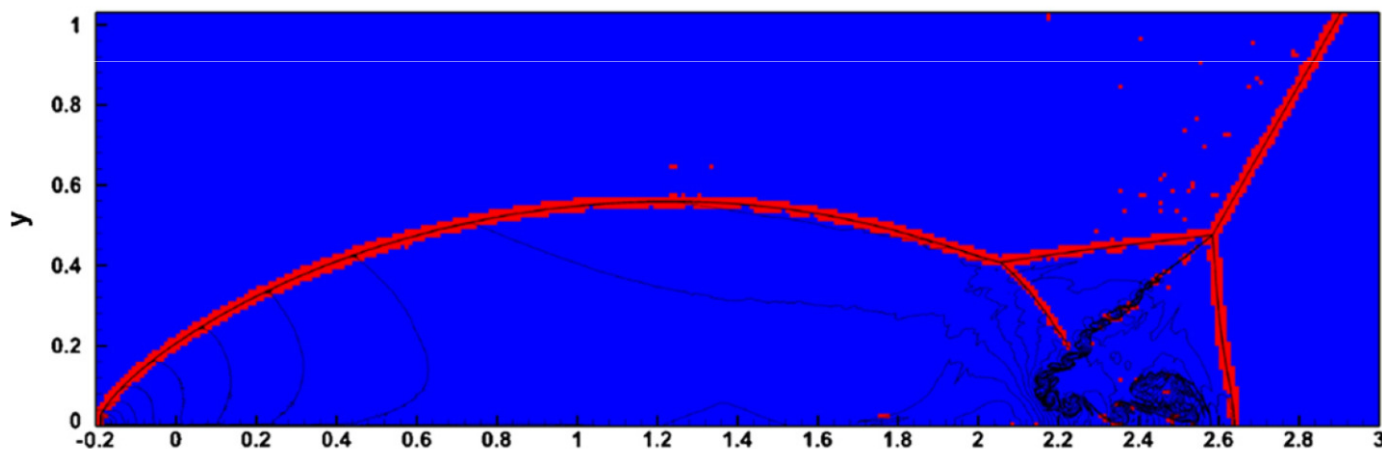
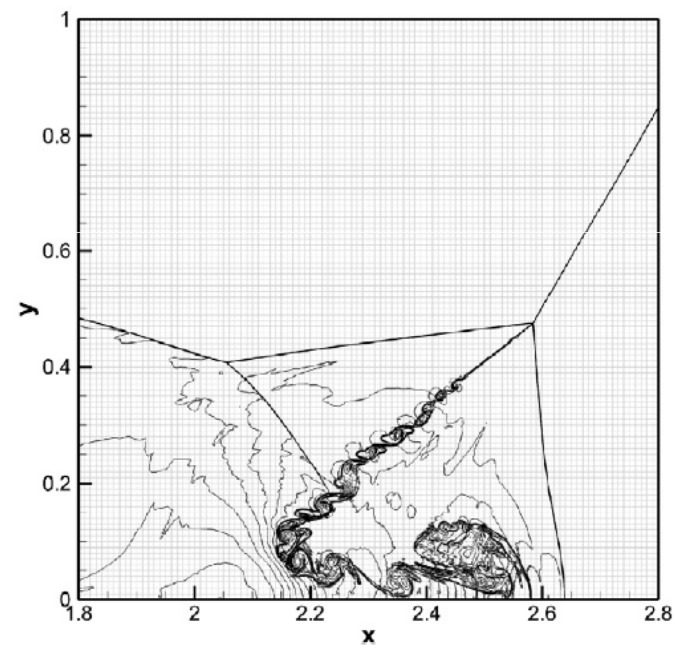
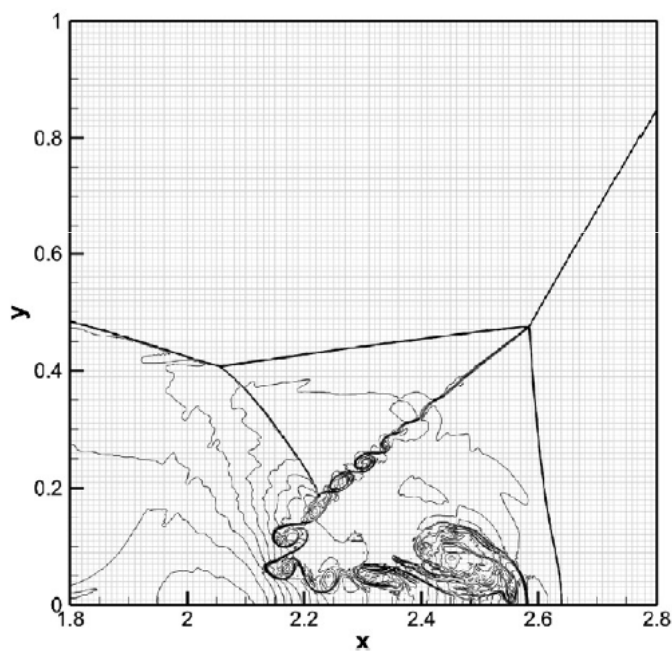
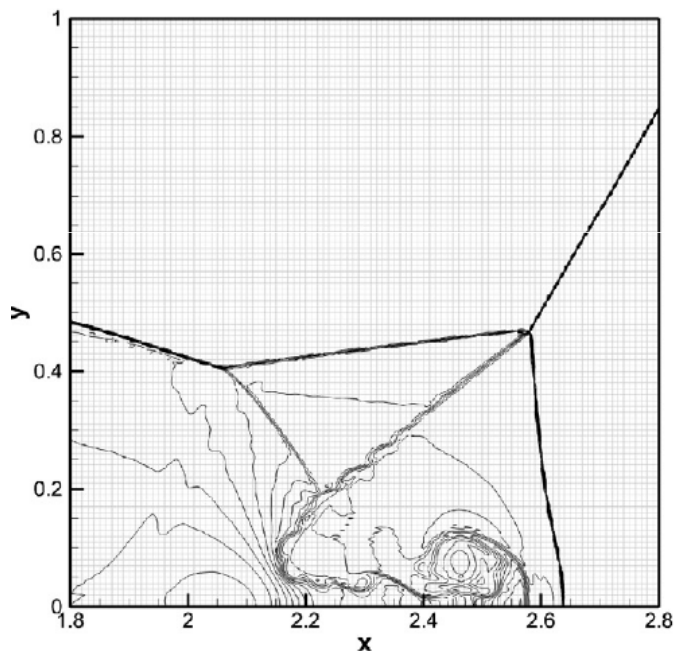
Limited cells (red),  
Unlimited cells (blue)

## ADER-DG-MOOD Results



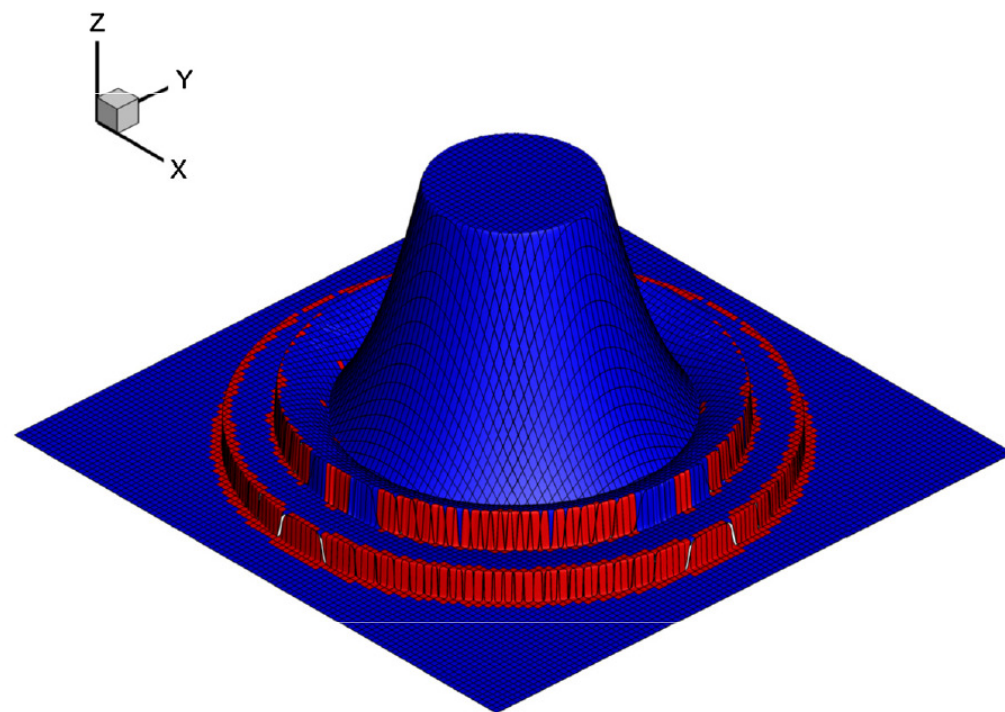
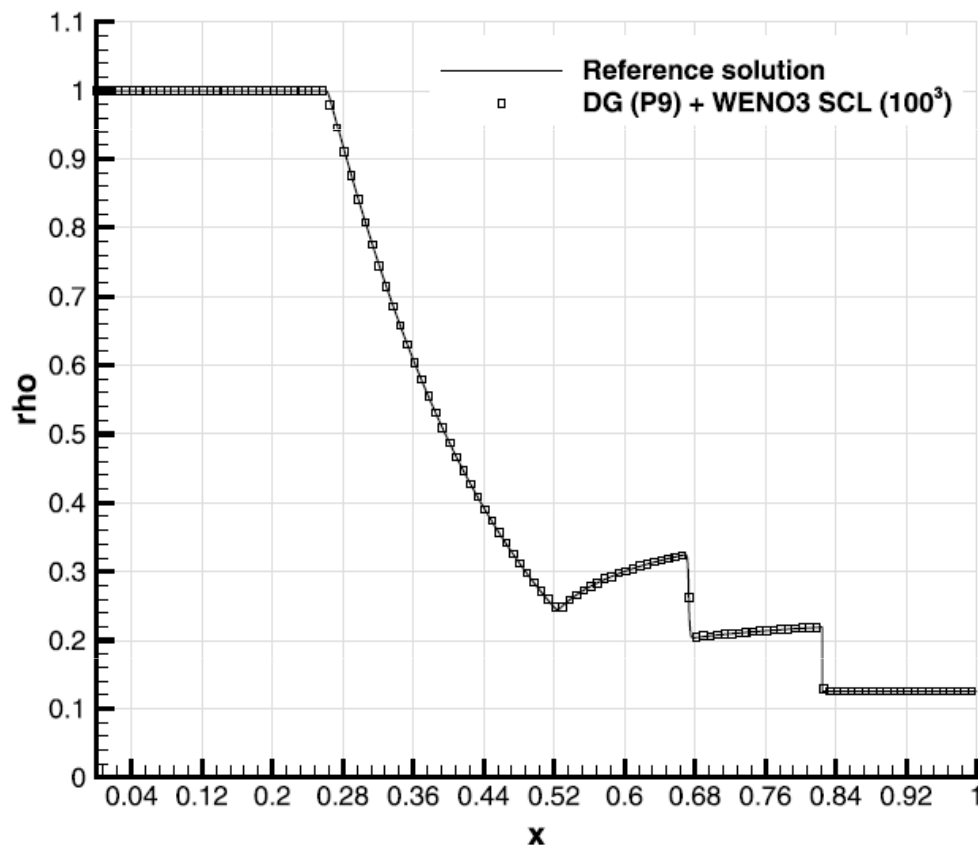
**Shock-density interaction problem of Shu & Osher  
40x5 cells (**N=9**). Unlimited cells (blue) and limited cells (red)**

## Double Mach Reflection Problem



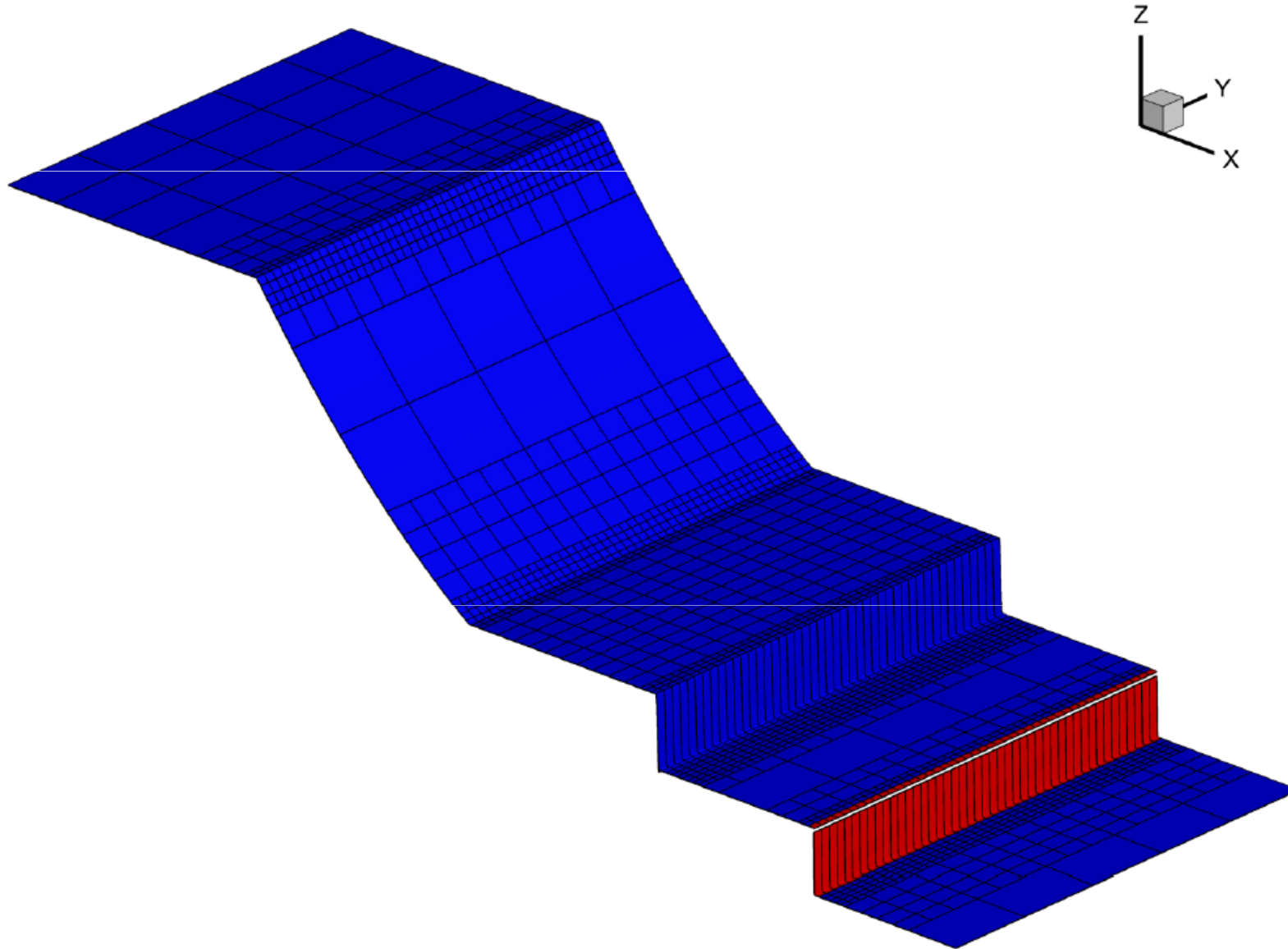


## 3D Spherical Explosion Problem

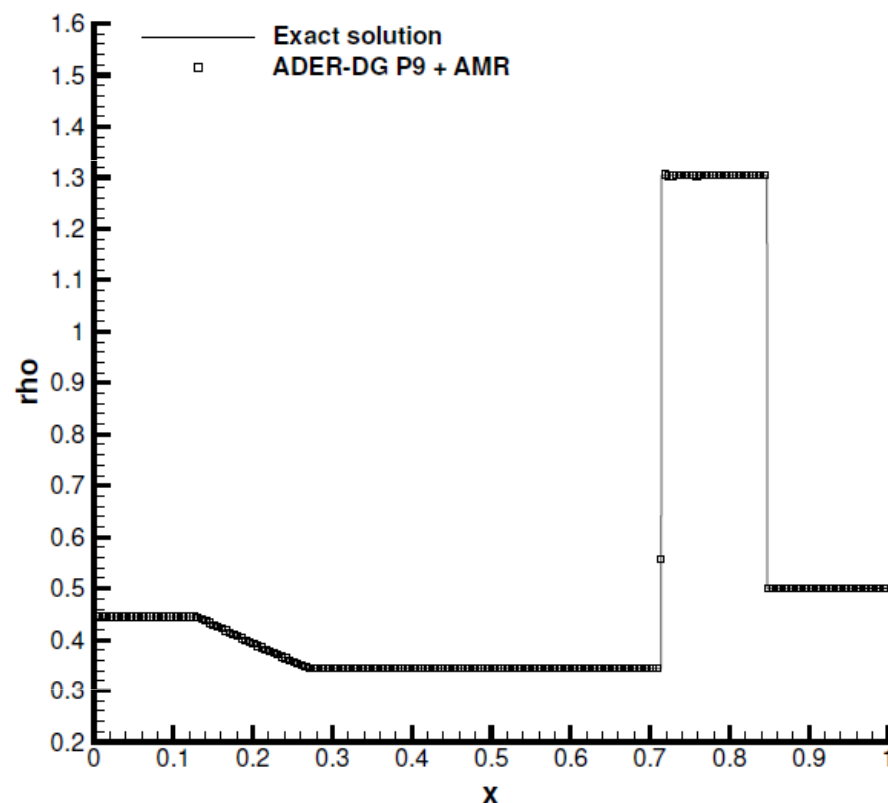
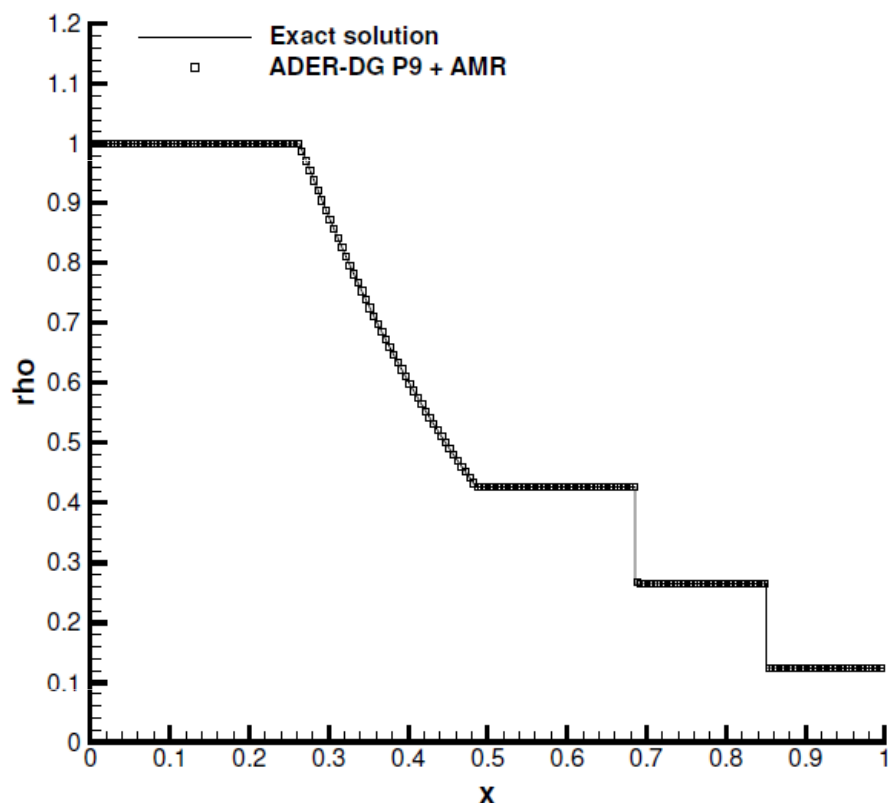


$100^3$  cells (**N=9**), corresponding to **10 billion** space-time degrees of freedom per time step. Unlimited cells (blue) and limited cells (red)

# Coupling of *a posteriori* subcell limiters for DG with AMR



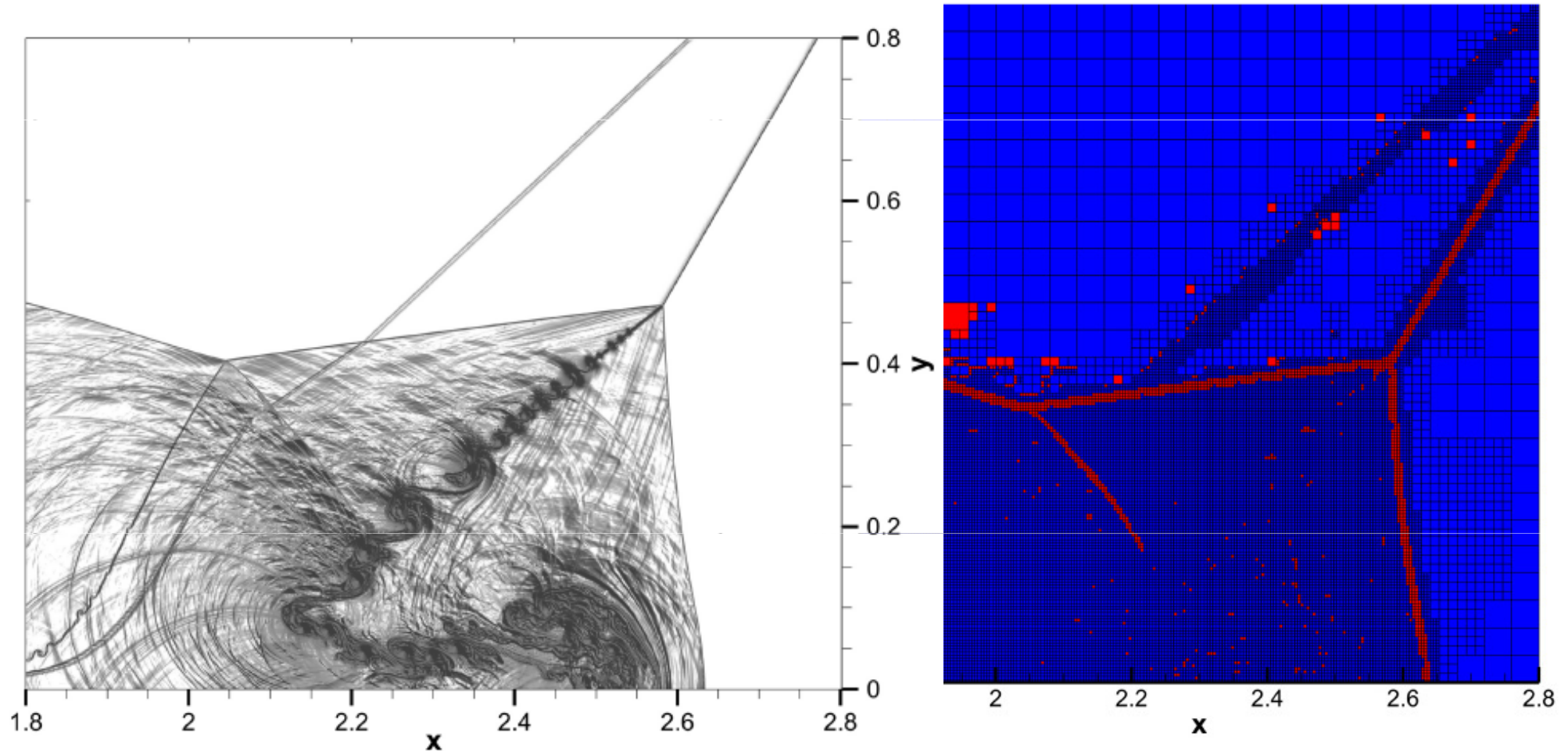
# Coupling of AMR with a posteriori subcell limiters for DG



ADER-DG ( $N=9$ ) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR) yields an unprecedented resolution of shocks and contact waves.

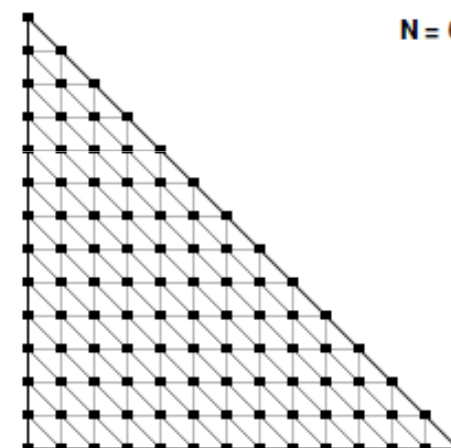
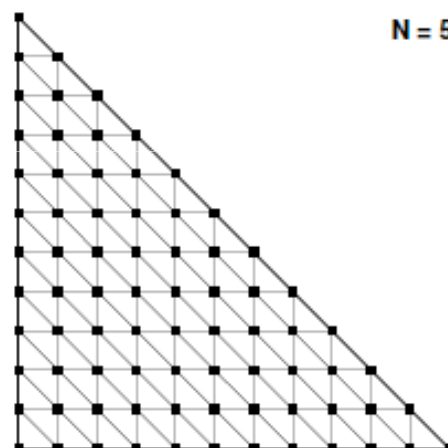
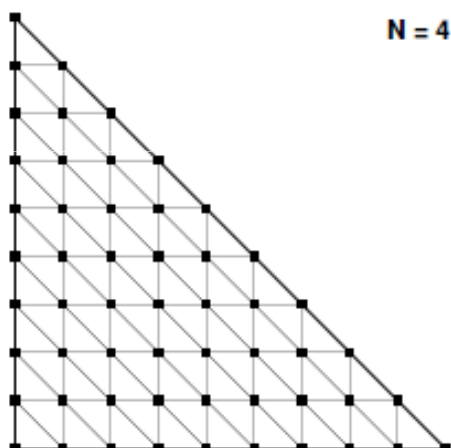
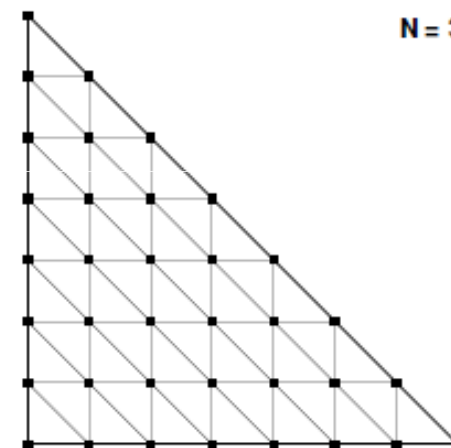
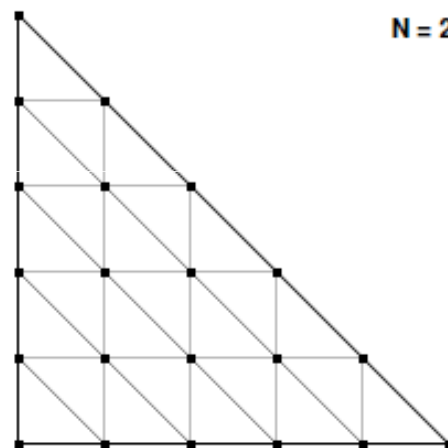
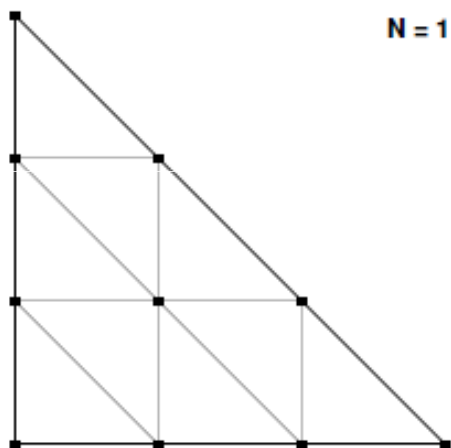


# Coupling of AMR with a posteriori subcell limiters for DG

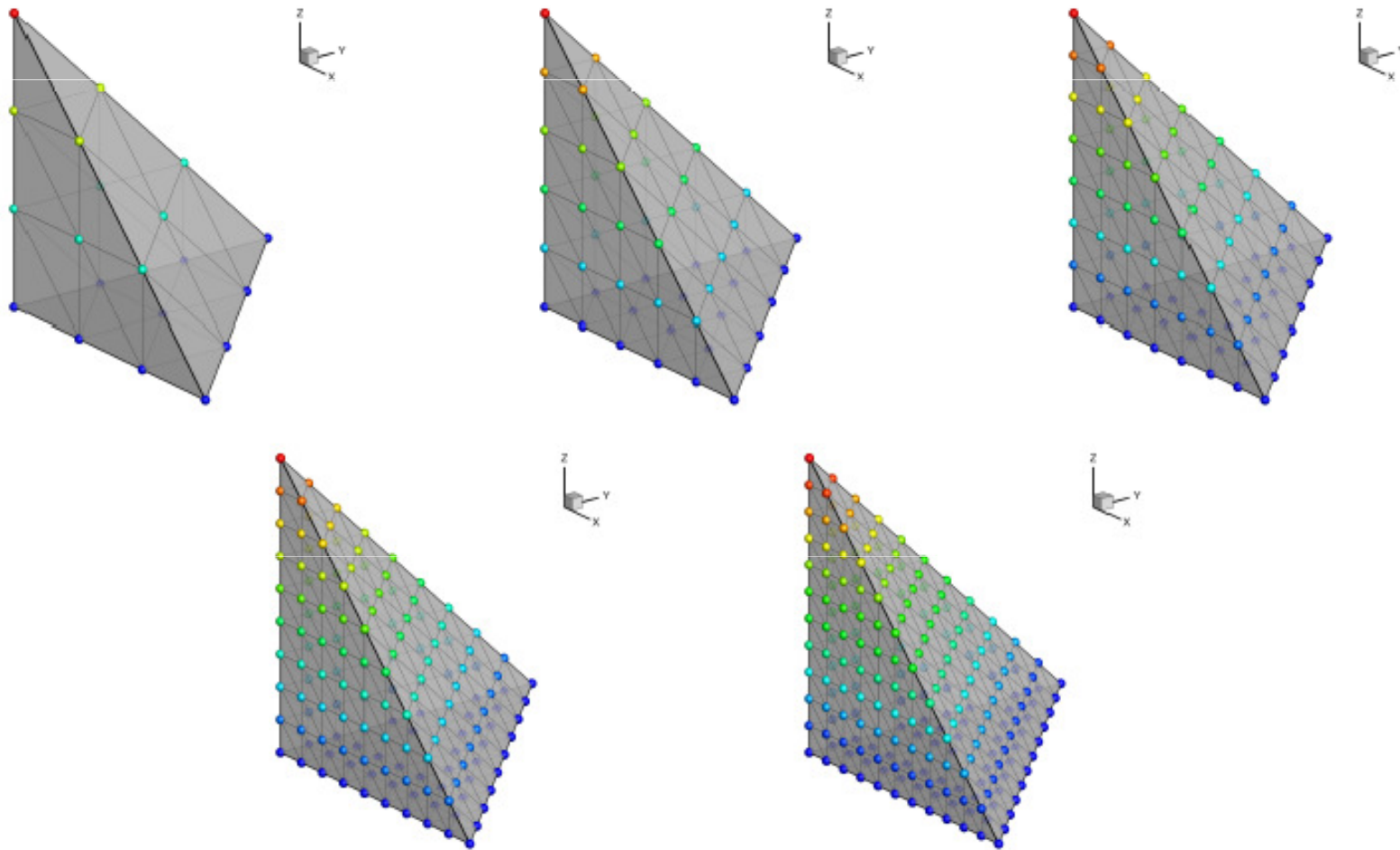


**Double Mach reflection problem using ADER-DG (N=9) with a posteriori ADER-WENO subcell limiter and space-time adaptive mesh refinement (AMR)**

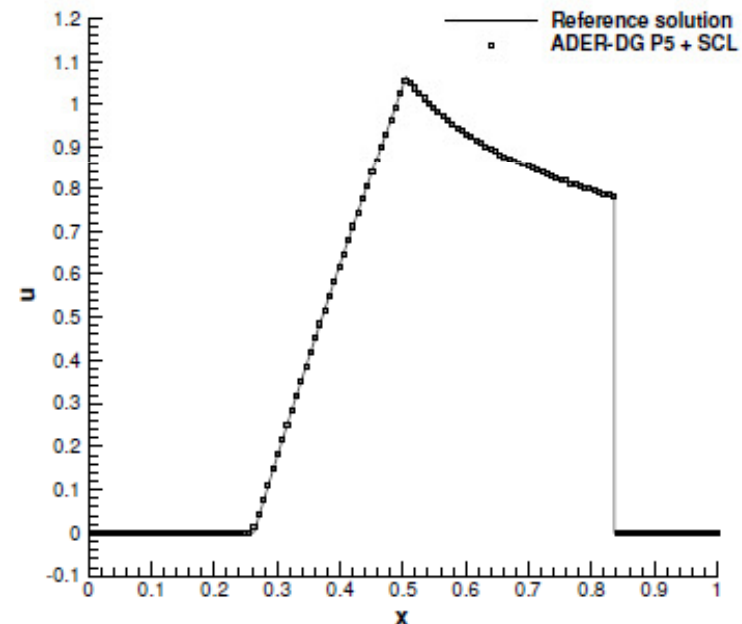
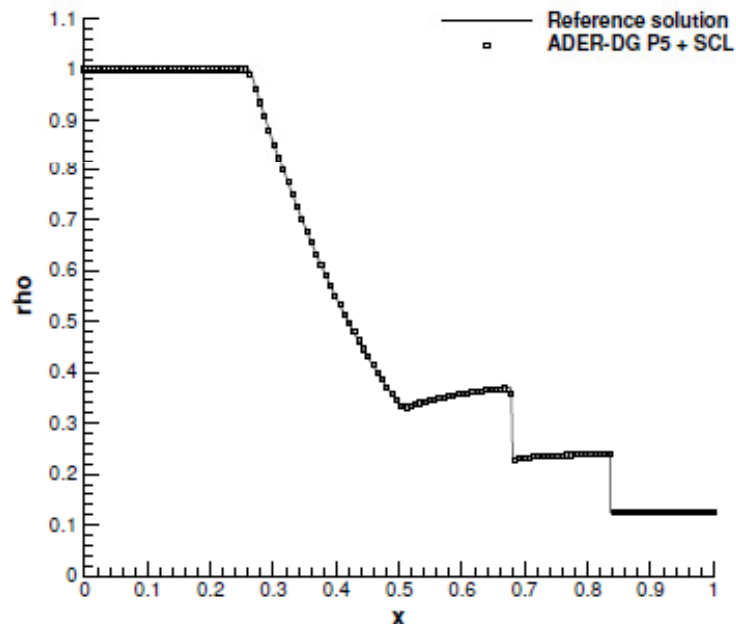
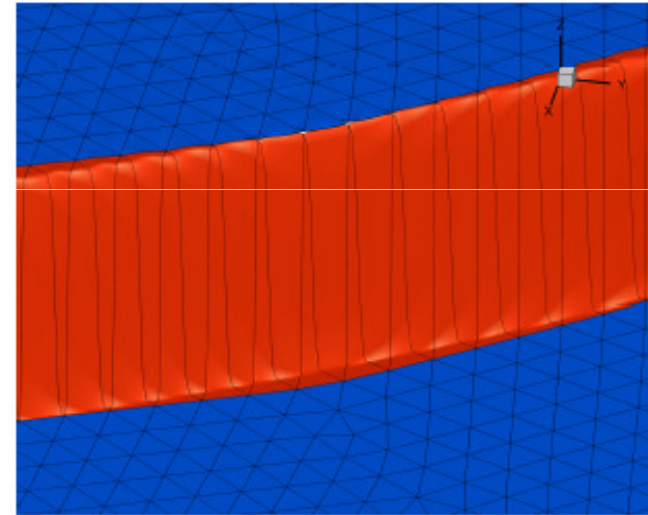
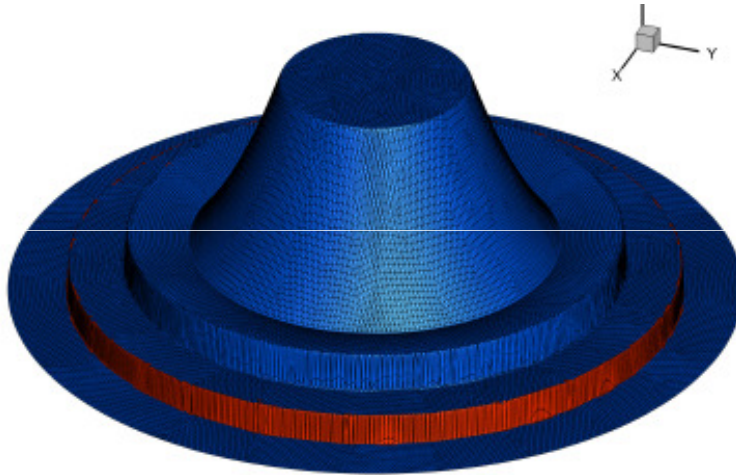
## Natural Extension to Unstructured Meshes



## Natural Extension to Unstructured Meshes

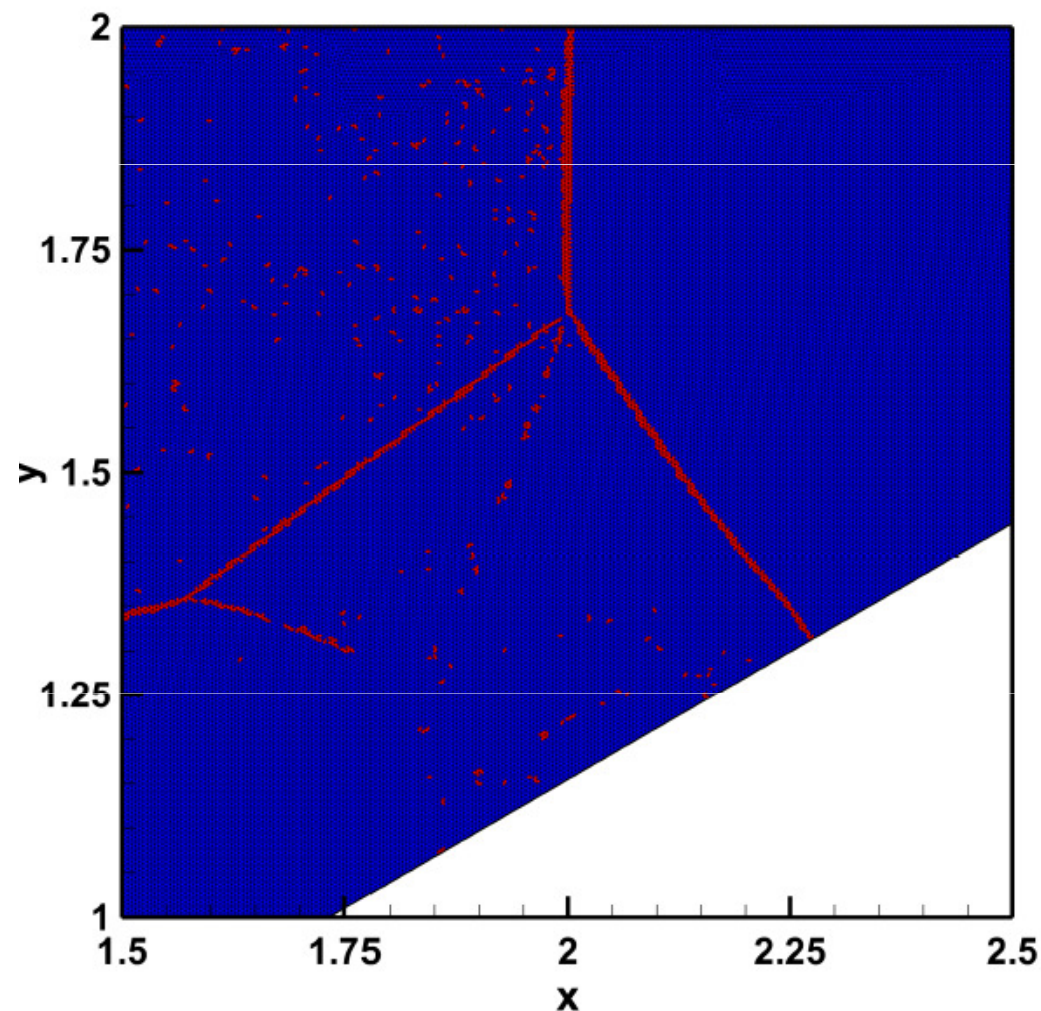
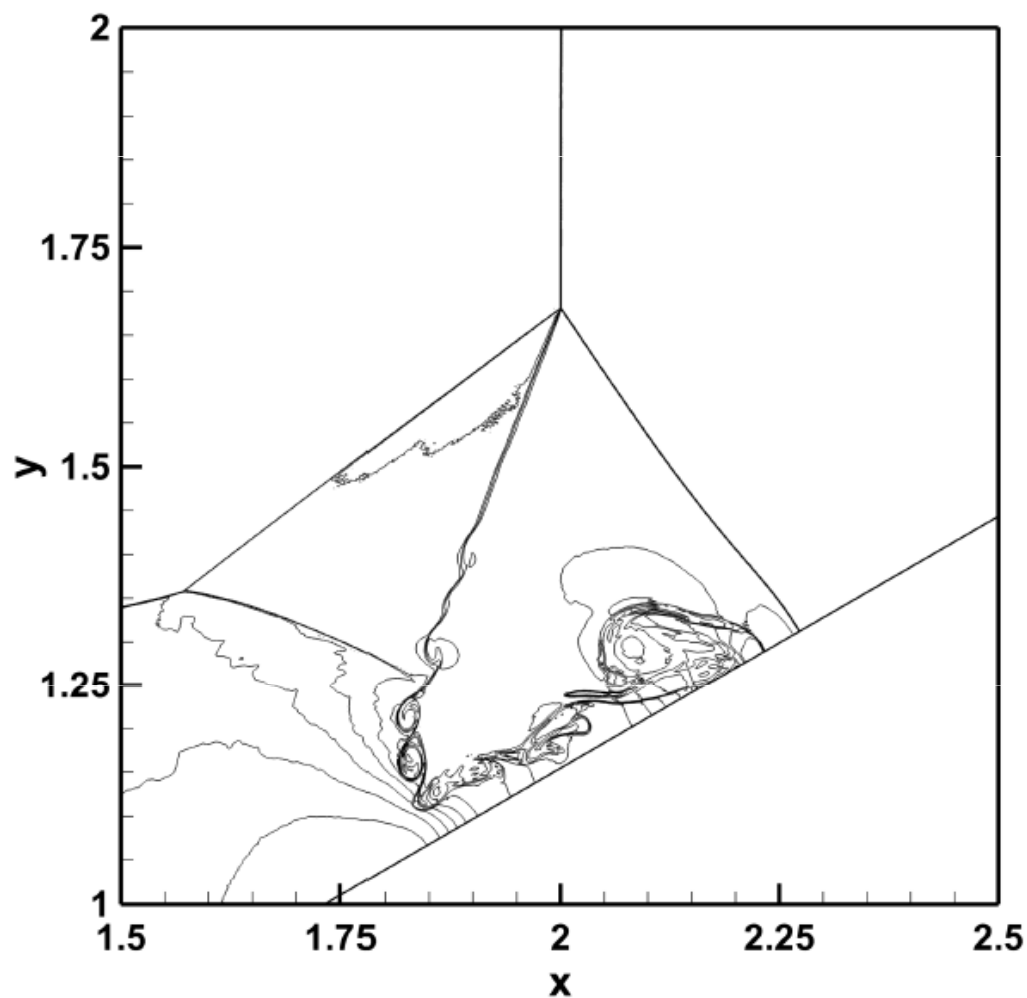


## Natural Extension to Unstructured Meshes

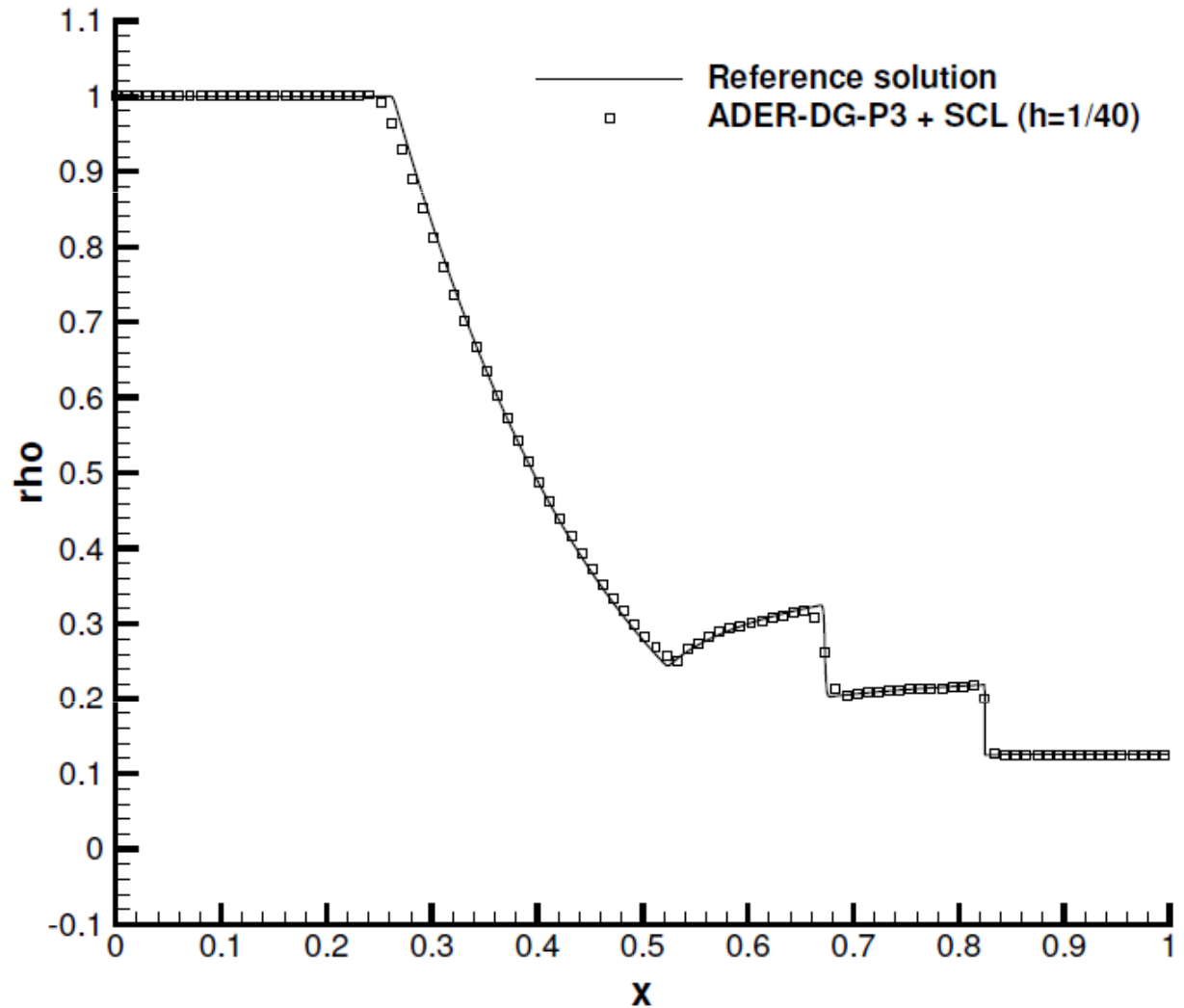
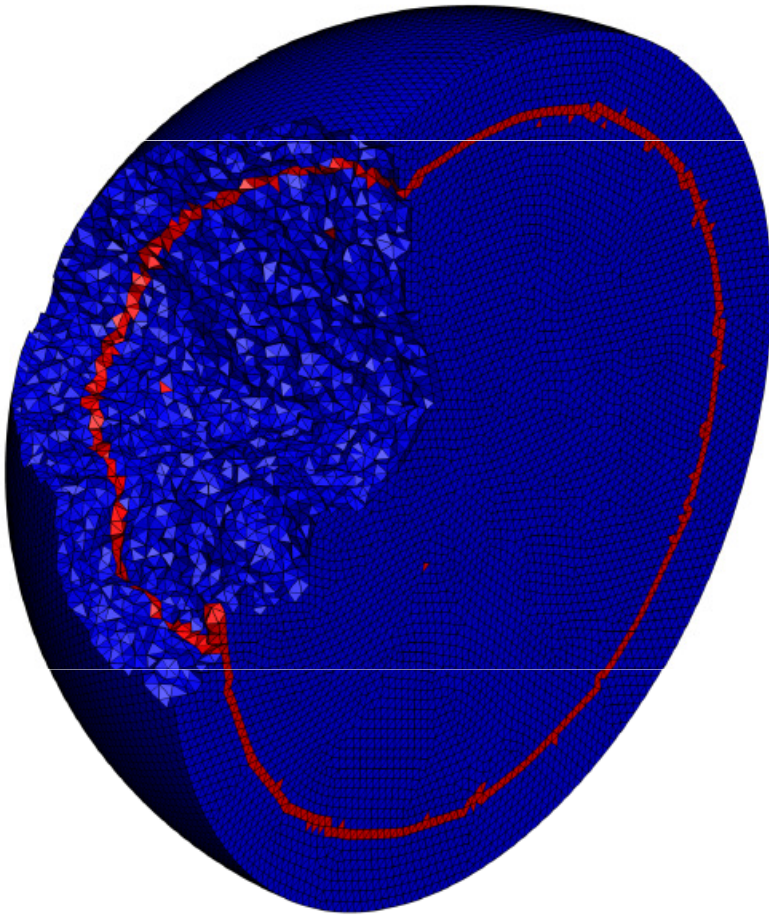




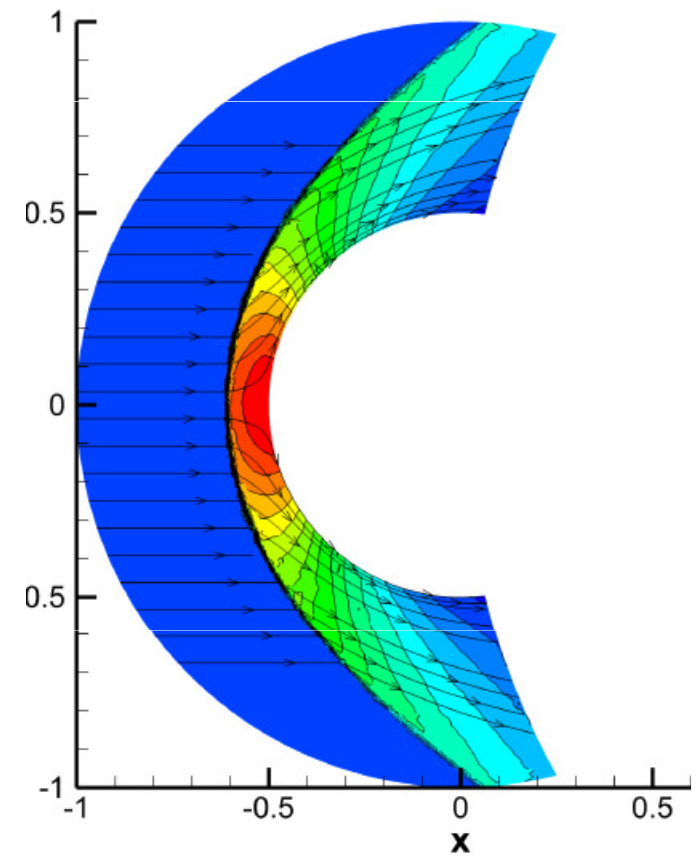
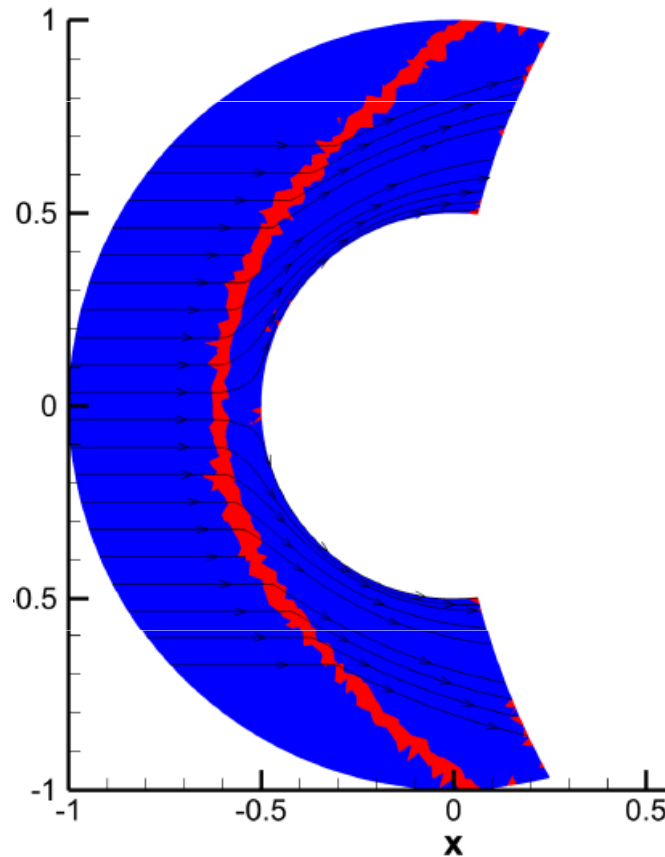
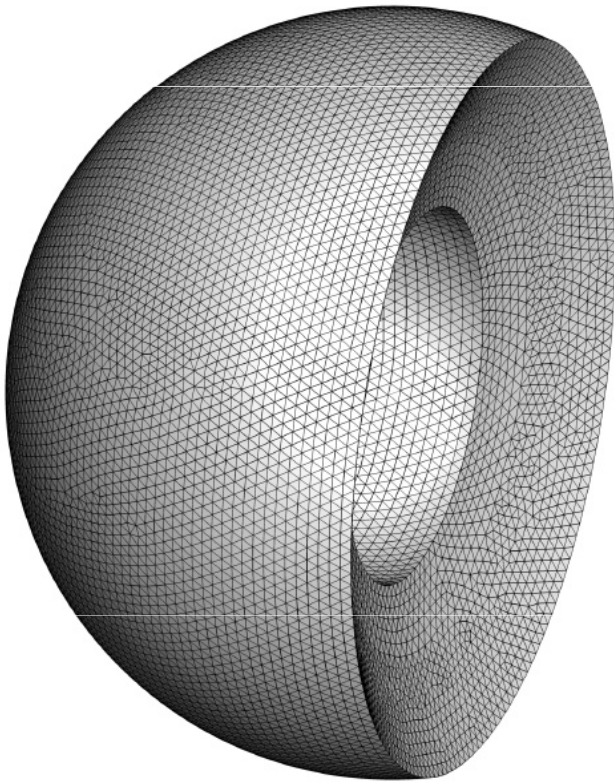
## Natural Extension to Unstructured Meshes



## Natural Extension to Unstructured Meshes



## Natural Extension to Unstructured Meshes





## Conclusions

- New, simple robust and accurate *a posteriori* subcell finite volume limiter for the discontinuous Galerkin finite element method
- High order fully discrete one-step ADER time discretization
- Available for uniform and space-time adaptive (AMR) Cartesian grids as well as for general triangular and tetrahedral unstructured meshes
- The *a posteriori* MOOD framework of Loubère, Clain and Diot has been found to be an ideal framework to devise a simple and robust limiter for DG schemes
- Why *a posteriori*: It is much simpler to **observe** (and cure) the occurrence of a troubled cell rather than to **predict** (and avoid) its occurrence from given data.
- Element-local **checkpointing** and solver **restarting** is even able to **cure** floating point errors (**NaN**, e.g. after division by zero)
- Future extension: **Lagrangian-type DG schemes** on unstructured ALE meshes