Mesh-adaptive computation of linear and non-linear acoustics

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1. Anisotropic mesh adaptation
2. Anisotropic goal-oriented mesh adaptation
3. Extension to unsteady
4. Applications to blast waves
5. Applications to linear acoustics

INRIA

Riemannian metric space: $(\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$

- Distance:

$$\text{Distance}(a,b) = \ell_{\mathcal{M}}(\mathbf{ab}) = \int_0^1 \sqrt{{}^t\mathbf{ab}\;\mathcal{M}(\mathbf{a}+t\mathbf{ab})\;\mathbf{ab}}\,\mathrm{d}t$$

- Complexity $\mathcal{C}$:

$$\mathcal{C}(\mathbf{M}) = \int_\Omega d(\mathbf{x})\,d\mathbf{x} = \int_\Omega \sqrt{\det(\mathcal{M}(\mathbf{x}))}\,d\mathbf{x}.$$

- Matrix writing:

$$\mathcal{M}(\mathbf{x}) = d^{\frac{2}{3}}(\mathbf{x})\,\mathcal{R}(\mathbf{x}) \begin{pmatrix} r_1^{-2/3}(\mathbf{x}) & & \\ & r_2^{-2/3}(\mathbf{x}) & \\ & & r_3^{-2/3}(\mathbf{x}) \end{pmatrix} {}^t\mathcal{R}(\mathbf{x}).$$

- **Main idea:** change the distance evaluation in the mesh generator
  [Vallet, 1992], [Casto-Diaz et Al., 1997], [Hecht et Mohammadi, 1997]

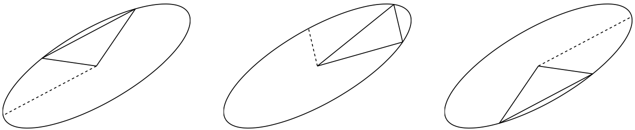
- **Fundamental concept:** **Unit mesh**

Adapting a mesh

$\Updownarrow$ Work in adequate Riemannian metric space

Generating a uniform mesh w.r.t. to $\mathcal{M}(\mathbf{x})$

$$\mathcal{H} \text{ unit mesh} \iff \forall \mathbf{e},\ \ell_{\mathcal{M}}(\mathbf{e}) \approx 1 \text{ and } \forall K,\ |K|_{\mathcal{M}} \approx \begin{cases} \sqrt{3}/4 & \text{in 2D} \\ \sqrt{2}/12 & \text{in 3D} \end{cases}$$



For any K which is unit for $\mathcal{M}$ and for all u quadratic positive form ($u(\mathbf{x}) = \frac{1}{2}\, {}^t\mathbf{x}\, H\, \mathbf{x}$):

$$\|u - \Pi_h u\|_{\mathbf{L}^1(K)} = \frac{\sqrt{2}}{240} \underbrace{\det(\mathcal{M}^{-\frac{1}{2}})}_{\textit{mapping}} \underbrace{\mathrm{trace}(\mathcal{M}^{-\frac{1}{2}}\, H\, \mathcal{M}^{-\frac{1}{2}})}_{\textit{anisotropic term}}$$

Continuous interpolation error:

$$\forall \mathbf{x} \in \Omega\,, \quad |u - \pi_{\mathcal{M}} u|(\mathbf{x}) \;=\; \frac{1}{10}\mathrm{trace}\big(\mathcal{M}(\mathbf{x})^{-\frac{1}{2}}\, |H(\mathbf{x})|\, \mathcal{M}(\mathbf{x})^{-\frac{1}{2}}\big)$$

equivalent because:

$$\frac{1}{10}\mathrm{trace}\big(\mathcal{M}(\mathbf{x})^{-\frac{1}{2}}\, |H(\mathbf{x})|\, \mathcal{M}(\mathbf{x})^{-\frac{1}{2}}\big) \;=\; 2\, \frac{\|u - \Pi_h u\|_{\mathbf{L}^1(K)}}{|K|}$$

for any K which is *unit* with respect to $\mathcal{M}(\mathbf{x})$.

We proposed a continuous mesh framework to solve this problem

Discrete	**Continuous**
Element K	Metric tensor $\mathcal{M}(\mathbf{x}_K)$
Mesh $\mathcal{H}$ of Ω_h	Riemannian metric space $\mathcal{M} = (\mathcal{M}(\mathbf{x}))_{\mathbf{x}\in\Omega}$
Number of vertices N_v	Complexity $\mathcal{C}(\mathcal{M}) = \int_\Omega \sqrt{\det(\mathcal{M}(\mathbf{x}))}\, d\mathbf{x}$
Linear interpolate $\Pi_h u$	**Continuous linear interpolate** $\pi_{\mathcal{M}} u$

We call multi-scale adaptation the minimisation of the L^p-norm, with $p < \infty$, of the continuous interpolation:

Find $\mathbf{M}_{opt} = (\mathcal{M}_{opt}(\mathbf{x}))_{\mathbf{x}\in\Omega}$ of complexity N such that

$$
\begin{aligned}
E_{\mathcal{M}_{opt}}(u) &= \min_{\mathcal{M}} \|u - \pi_{\mathcal{M}} u\|_{\mathcal{M},\mathbf{L}^p(\Omega)} \\
&= \min_{\mathcal{M}} \left(\int_{\Omega} |u(\mathbf{x}) - \pi_{\mathcal{M}} u(\mathbf{x})|^p \,\mathrm{d}\mathbf{x} \right)^{\frac{1}{p}}
\end{aligned}
$$

A well-posed problem solved by a calculus of variations.

Optimal metric

$$\mathcal{M}_{\mathbf{L}^p} \;=\; \underset{①}{D_{\mathbf{L}^p}} \;\; \underset{②}{(\det|H_u|)^{\frac{-1}{2p+3}}} \;\; \underset{③}{\mathcal{R}_u^{-1}} \;\; \underset{④}{|\Lambda|} \;\; \mathcal{R}_u$$

1. Global normalization: to reach the constraint complexity N

$$D_{\mathbf{L}^p} = N^{\frac{2}{3}} \left(\int_\Omega (\det|H_u|)^{\frac{p}{2p+3}} \right)^{-\frac{2}{3}} \quad \text{and} \quad D_{\mathbf{L}^\infty} = N^{\frac{2}{3}} \left(\int_\Omega (\det|H_u|)^{\frac{1}{2}} \right)^{-\frac{2}{3}}$$

2. Local normalization: sensitivity to small solution variations, depends on $\mathbf{L}^p$ norm chosen
3. Optimal anisotropy directions based on Hessian eigenvectors
4. Diagonal matrix of anisotropy strengths, defined from the absolute values of Hessian eigenvalues

Fixed point algorithm

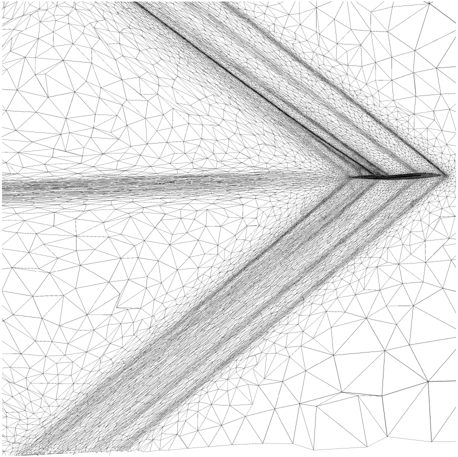
- Compute flow
- Compute metric field
- Build new mesh
- Interpolate old data on new mesh

Background and properties:

[Castro Diaz *et al.*, 1997], [Habashi *et al.*, 2000], [Frey and Alauzet, 2005], ...

- Genericity, does not depend on the PDE and on the numerical scheme
- Anisotropy easily deduced
- The multiscale (*i.e.* L^p) version provides an optimal mesh without neglecting weaker details.

An example: supersonic steady flow around an aircraft.



Objectif

Deriving the best mesh to observe a given functional $j(w) = (g, w)$ depending of the solution w of a PDE and enough regular to be observed through its Jacobian g.

How?

Control of the approximation error on the output functional : $j(w) - j(w_h)$.

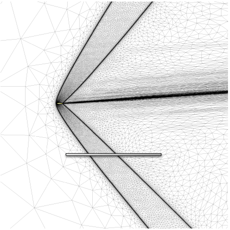
Exemples

- vorticity in wake $j(\mathbf{w}) = \int_\gamma \|\nabla \wedge (\mathbf{u} - \mathbf{u}_\infty)\|_2^2 \, \mathrm{d}\gamma$
- drag, lift: use to quantify the performance of a design , etc...

Background:

[Becker-Rannacher],[Giles-Pierce],[Venditti-Darmofal, 2002],[Rogé-Martin, 2008],. . .

- Explicit use of the PDE
- Strong dependency on the numerical scheme
- Anisotropy hard to prescribe



- Given a functional $j(w)$
- We only know w_h
- How to control $j(w) - j(w_h)$

Continuous and discrete equations

$$(\Psi(w), \phi) = 0 \quad \text{and} \quad (\Psi_h(w_h), \phi_h) = 0$$

Continuous and discrete adjoint equations

$$(\frac{\partial \Psi}{\partial w}(w)\phi, w^*) = (g, \phi) \quad \text{and} \quad (\frac{\partial \Psi_h}{\partial w}(w_h)\phi_h, w_h^*) = (g, \phi_h)$$

Adjoint estimation

- Dual formula [Giles et Süli, 2002]

$$j(w) - j(w_h) \approx (g, w - w_h) \quad = \quad \underbrace{-(w^*, \Psi(w_{\color{red}h}))}_{\textit{A posteriori}} \quad = \quad \underbrace{(w^*_{\color{red}h}, \Psi_{\color{red}h}(w))}_{\textit{A priori}}$$

A priori error estimation [A. Loseille and A. Dervieux and F. Alauzet, Fully anisotropic goal-oriented mesh adaptation for 3D steady Euler equations, JCP, 2010]

$$
\begin{aligned}
j(w) - j(w_h) &= \underbrace{(g, w - w_h)}_{\textit{Approximation error}} = \underbrace{(g, w - \Pi_h w)}_{\textit{Interpolation error}} + \underbrace{(g, \Pi_h w - w_h)}_{\textit{Implicit error}} \\
&= \big((\Psi_h - \Psi)(w), w_h^*\big) + R_3
\end{aligned}
$$

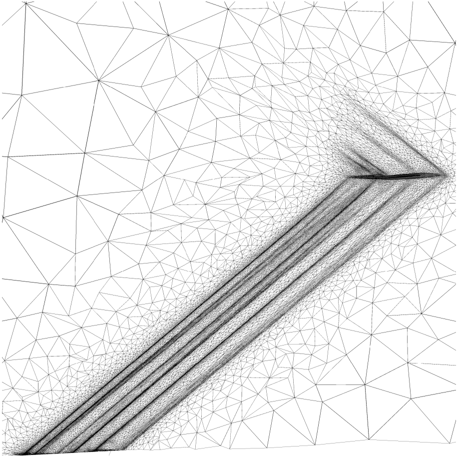
- Search for continuous model $E(\mathcal{M})$ to evaluate $(\Psi_h - \Psi)(w)$.
- Find $\mathcal{M}$ that minimises $\big(E(\mathcal{M}), w^*\big)$.

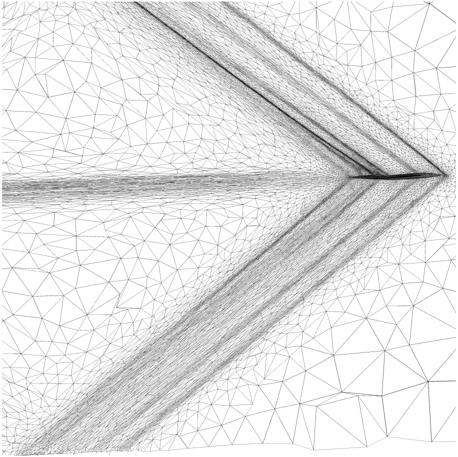
Application to sonic boom :

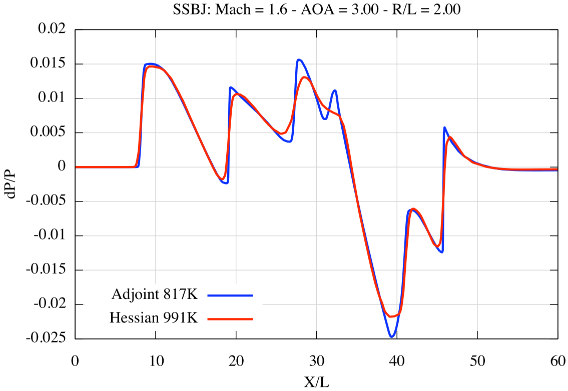
- Adjoint functional :

$$j(W) = \int_{\gamma} \left(\frac{p - p_{\infty}}{p_{\infty}} \right)^2 \mathrm{d}\gamma$$

- Adaptation variable : Mach number







Even close to the aircraft (2 lengths), the adjoint-based adaptation strongly supersedes the multiscale method.

Problematics:

- Evolution of physical phenomena in time.
- One may need a good prediction of solution evolution into the whole computational domain. In this case, the unsteady multiscale method need be applied. We refer to Alauzet *et al.* JCP (2007).
- A target observation can be specified: the goal oriented version is needed.
- We neglect time-discretisation errors in the present study.

$$(\Psi(W),\Phi) = \int_Q \Phi \;\partial_t W \,\mathrm{d}Q + \int_Q \Phi \;\nabla.\mathcal{F}(W)\,\mathrm{d}Q - \int_\Sigma \Phi \;\hat{\mathcal{F}}(W)\,\mathrm{d}\Sigma$$

$$(\Psi_h(W),\Phi_h) = \int_Q \Phi_h \;\Pi_h\partial_t W \,\mathrm{d}Q + \int_Q \Phi_h\nabla.\Pi_h\mathcal{F}(W)\,\mathrm{d}Q - \int_\Sigma \Phi_h\Pi_h\hat{\mathcal{F}}(W)\,\mathrm{d}\Sigma$$

with $Q = \Omega \times]0,T[$, $\Sigma = \partial\Omega \times]0,T[$.

Let:

$$j(w) = (g,w)_Q$$

$$j(w) - j(w_h) \approx \int_Q W^* \;(\partial_t W_h - \partial_t W + \nabla.\mathcal{F}_h(W) - \nabla.\mathcal{F}(W))\,\mathrm{d}Q + \mathrm{BT}$$

$$= \int_Q W^* \;(\partial_t W_h - \partial_t W)\,\mathrm{d}Q + \int_Q \nabla.W^* \;(\mathcal{F}(W) - \mathcal{F}_h(W))\,\mathrm{d}Q + \mathrm{BT}$$

$$= \int_Q W^* \;(\Pi_h\partial_t W - \partial_t W)\,\mathrm{d}Q + \int_Q \nabla.W^* \;(\mathcal{F}(W) - \Pi_h\mathcal{F}(W))\,\mathrm{d}Q + \mathrm{BT}$$

Boundary integrals ("BT") are tranformed in a similar manner.

Solve this problem in the continuous framework

Find $\mathbf{M}_{opt} = (\mathcal{M}_{opt}(\mathbf{x}))_{\mathbf{x}\in Q}$ of complexity N such that

$$E(\mathcal{M}_{opt}) = \min_{\mathcal{M}} \Big(\int_Q W^* \ (\pi_{\mathcal{M}} W_t - W_t)\,\mathrm{d}Q + \\ + \int_Q \nabla . W^* \,(\mathcal{F}(W) - \pi_{\mathcal{M}}\mathcal{F}(W))\,\mathrm{d}Q + \mathrm{BT} \ \Big)$$

A calculus of variations gives

$$\mathcal{M}_{opt} = \mathcal{M}_{opt}^{\mathbf{L}^1} \left(\sum_{i=1}^{5} (|W_h^*(W_i)|\,|H(W_{i,t})| + \sum_{j=1}^{3} |\nabla_{x_j} W_h^*(W_i)|\,|H(\mathcal{F}_{x_j}(W_i))|) \right)$$

Discrete State System and functional:

$$\Psi_h^{n+1}(W^n, W^{n+1}, \phi^n) = 0 \Leftrightarrow W = W_{sol}$$

$$j = J(W_{sol})$$

Discrete Adjoint State System writes:

$$W^{*,N} = \left(\frac{\partial \Psi_h^N}{\partial W^N}\right)^{-T} \left(\frac{\partial J}{\partial W^N}\right)^T$$

$$W^{*,n} = \left(\frac{\partial \Psi_h^n}{\partial W^n}\right)^{-T} \left[\left(\frac{\partial J}{\partial W^n}\right)^T - \left(\frac{\partial \Psi_h^{n+1}}{\partial W^n}\right)^T W^{*,n+1}\right] \forall\ n = \overline{N-1, 0}$$

$\Longrightarrow$ Adjoint State is computed backwards in time.

Adjoint is advanced forward in time:

- Computing $W^{*,n}$ from the adjoint state $W^{*,n+1}$ needs the knowledge of states W^n, W^{n+1} .
- Higher-Order scheme with intermediate storage (like explicit Runge-Kutta schemes) demands even more storage/recompute effort

Our approach:

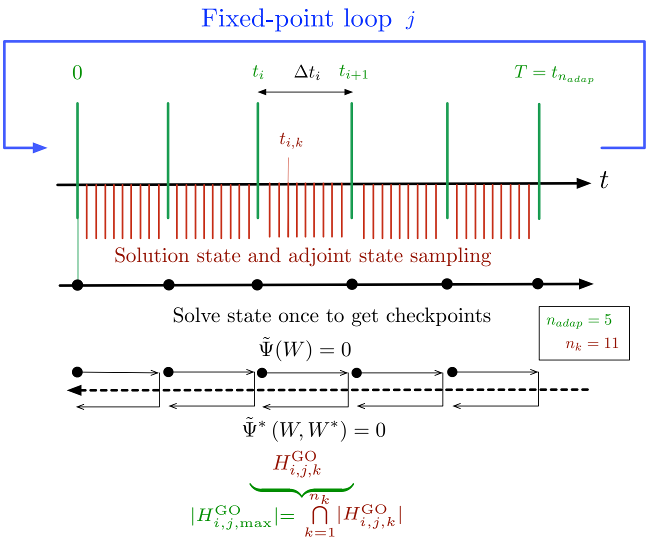
- Storage of the solution on checkpoints $\Longrightarrow$ forward/backward computation only between two checkpoints.
- Interpolate Adjoint states between two adaptation sub-intervals.

Optimal Metric computation needs:

- Adjoint state : W^* (computed backwards in time)
- Adjoint state gradient : $\nabla\ W^*$
- Hessian of the Euler fluxes : $H(\mathcal{F}(W))$
- Hessian of time derivative: $H(W_t)$

Continuous states $\Longleftarrow$ approximated by the discrete ones

Gradients and Hessians $\Longleftarrow$ derivative recovery (L^2 -projection)


Fixed-point loop j
0
t_i
Δt_i
t_{i+1}
$T = t_{n_{adap}}$
$t_{i,k}$
t
Solution state and adjoint state sampling
Solve state once to get checkpoints
$n_{adap} = 5$
$n_k = 11$
$\tilde{\Psi}(W) = 0$
$\tilde{\Psi}^* (W, W^*) = 0$
$H^{GO}_{i,j,k}$
$|H^{GO}_{i,j,\max}| = \bigcap_{k=1}^{n_k} |H^{GO}_{i,j,k}|$

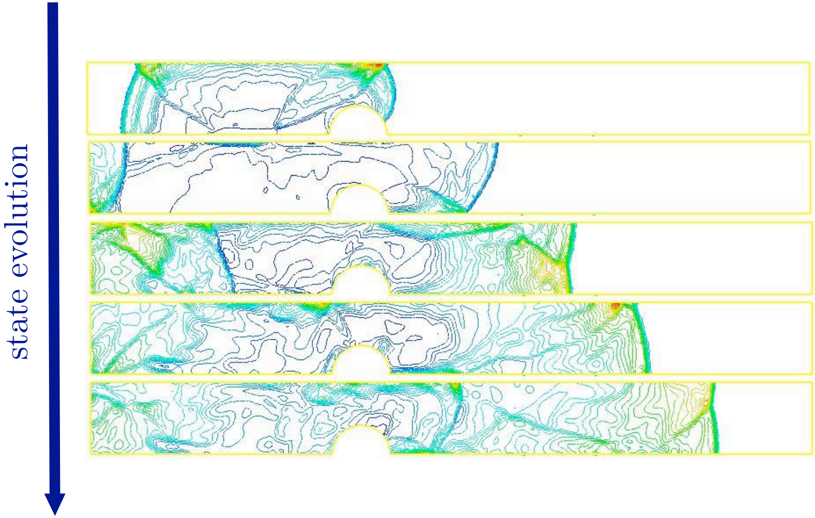
Blast-like initialisation inside a circle of radius $r_0 = 0.15$ around $x_0 = (1.2, 0.0)$, given by: $\rho = 10.0$, $v = (0, 0)$ and $e = 25.0$.

The cost function j was the impulse over the target surface S in Figure below:

$$j(W) = \frac{1}{2} \int_S (p - p_\infty)^2 ds.$$

Figure: Channel flow 2D mesh

4. APPLICATION TO A BLAST WAVE



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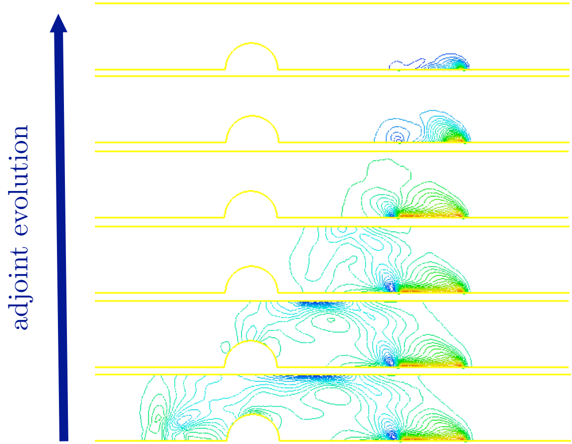
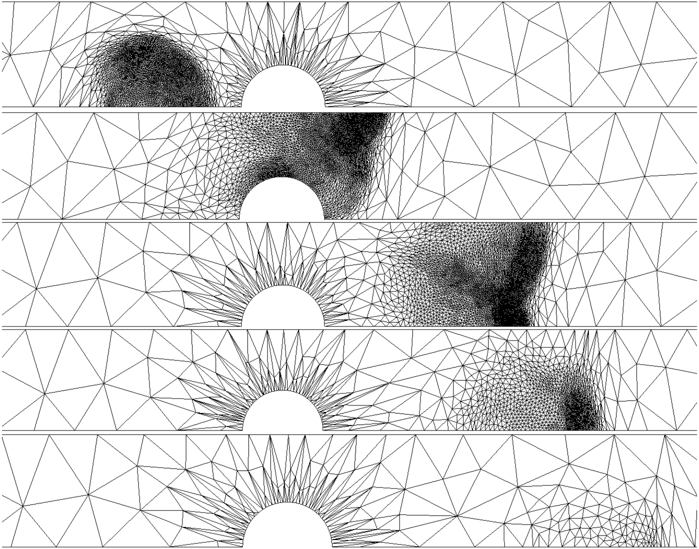
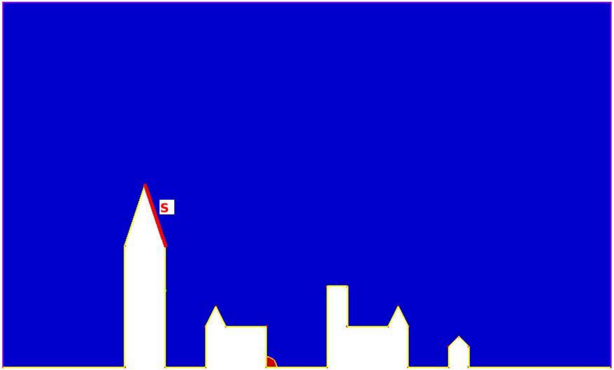


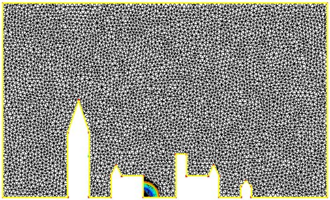


Second Example

Nonlinear "blast" wave.



Second Example, results



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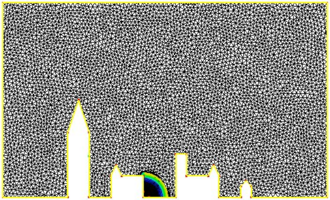

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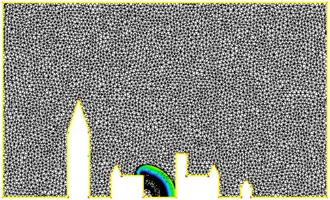

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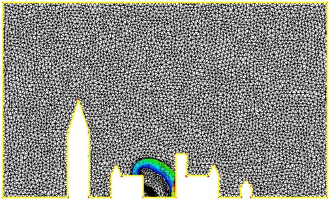

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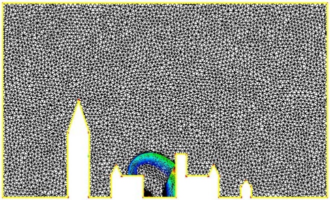

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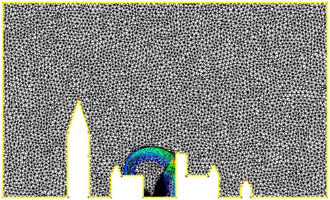

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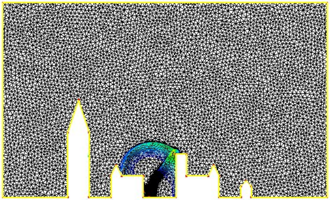

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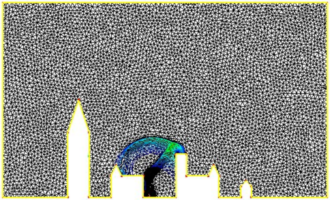

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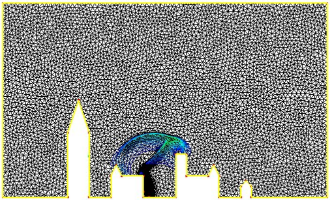

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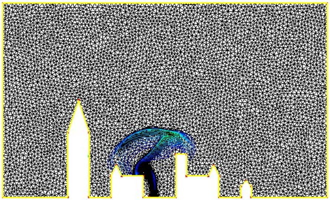

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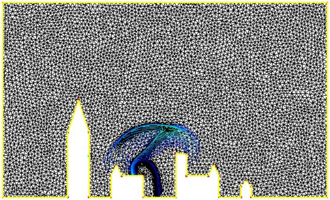

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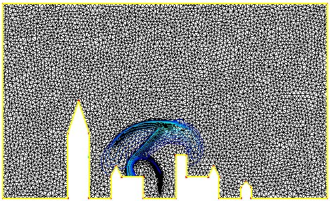

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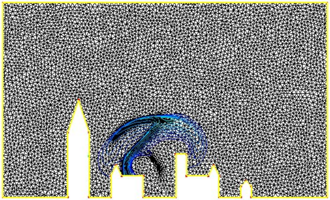

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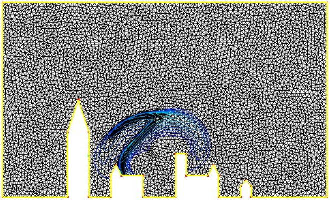

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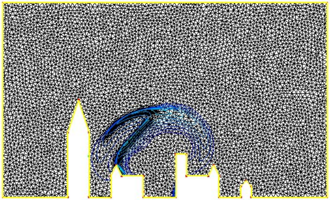

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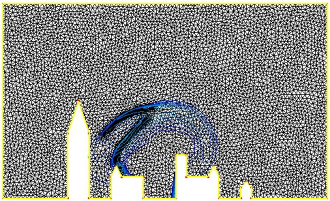

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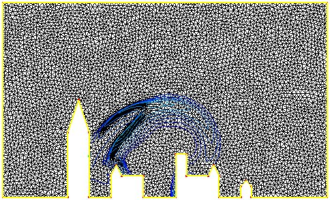

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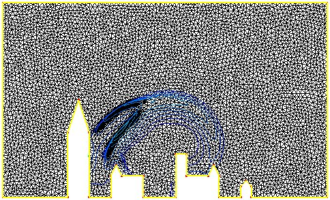

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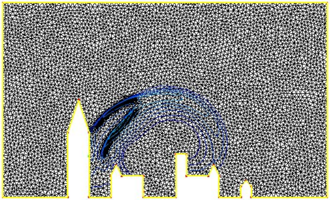

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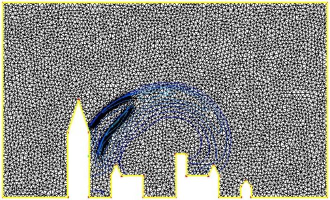

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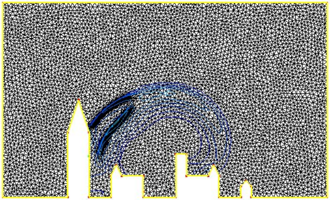
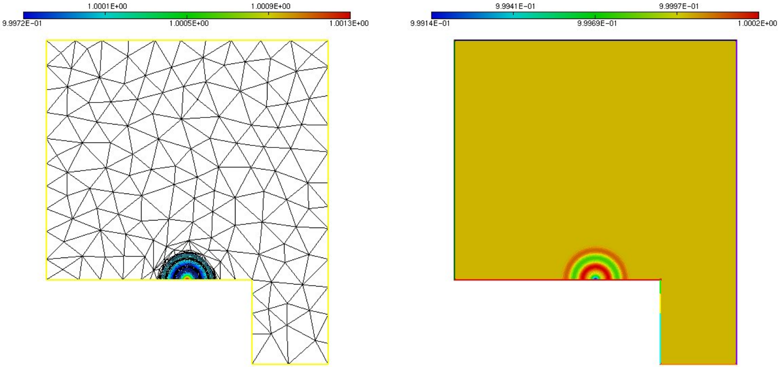


- For acoustic analysis, the use of anisotropic meshes seems less mandatory.
- Uniform meshes allow a higher accuracy with lower cost per node, but need good absorbing boundaries.
- For a particular family of problems, noise emission and noise observation ("micro") are localised in a small portion of the domain and much resolution can be useless.
- In that case, the goal-oriented formulation helps focalising the mesh effort on the propagation from source to micro.

Comparison between Goal-Oriented approach and multiscale L^p:



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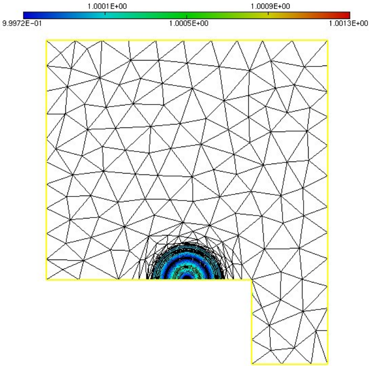

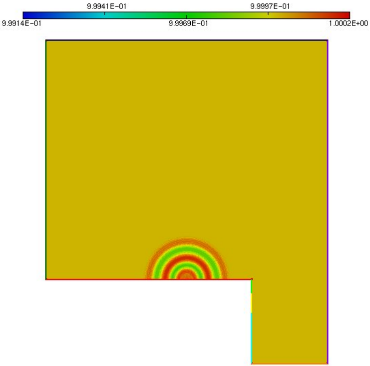

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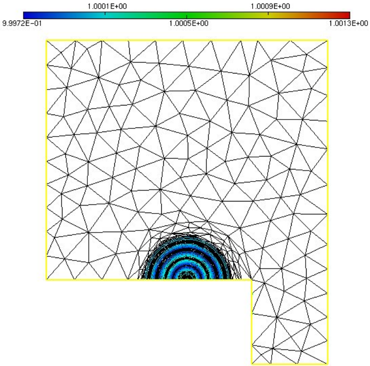

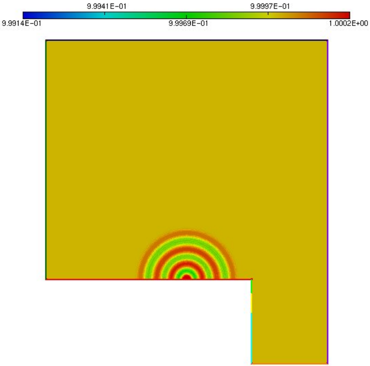

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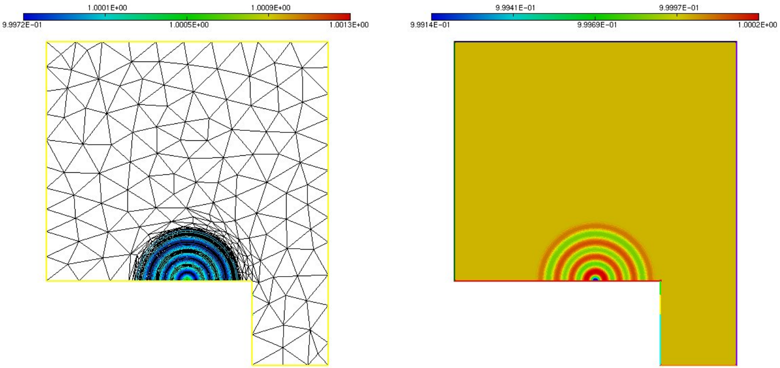

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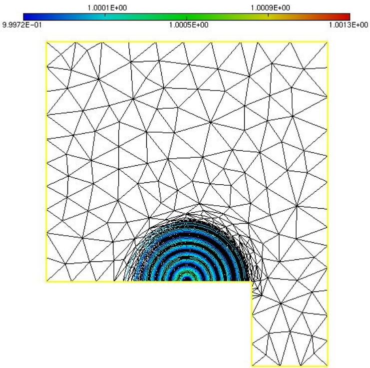

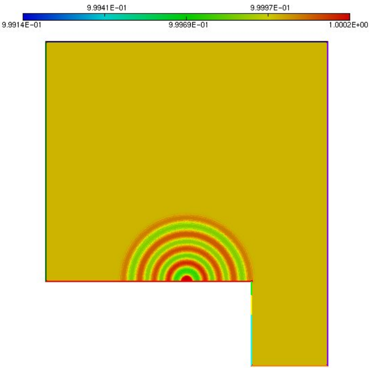

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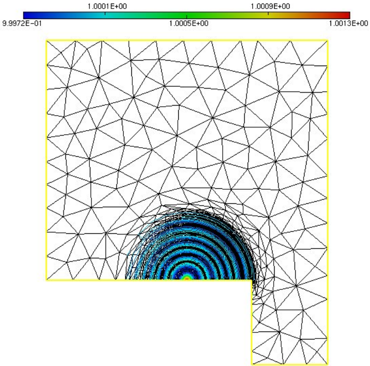

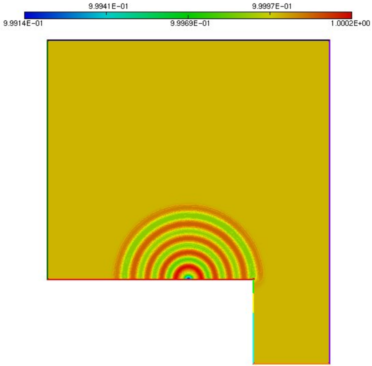

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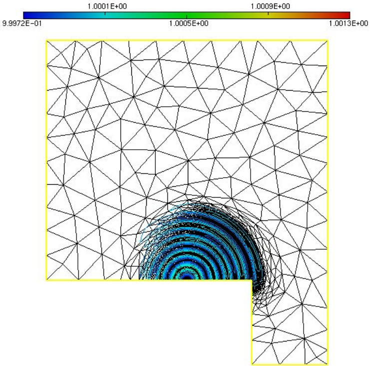

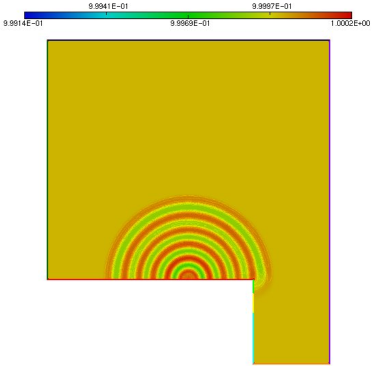

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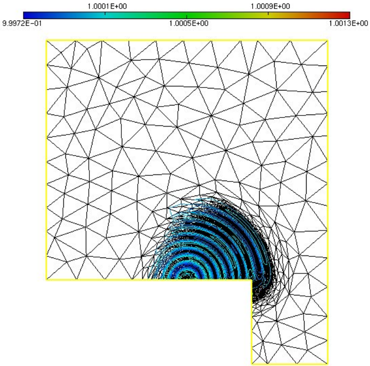

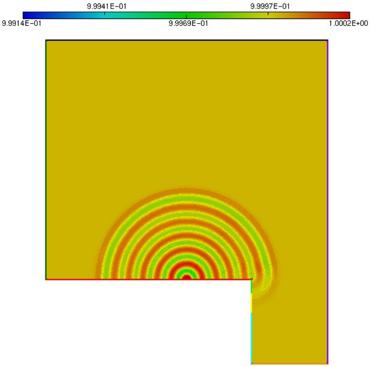

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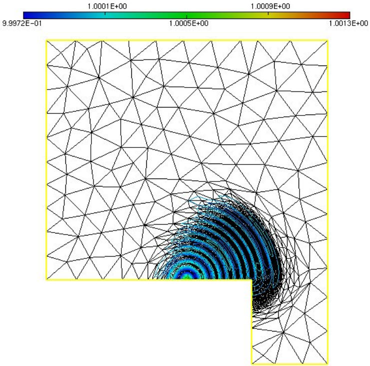

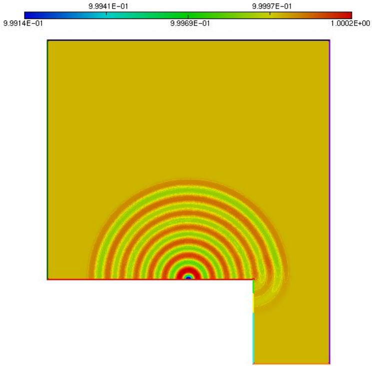

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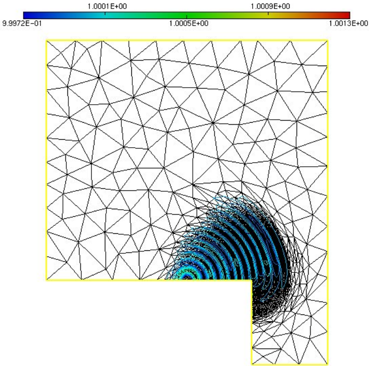

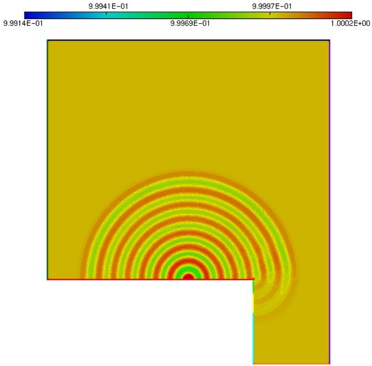

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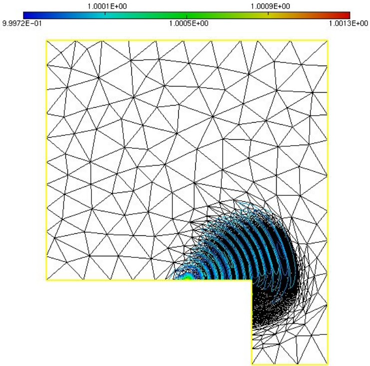

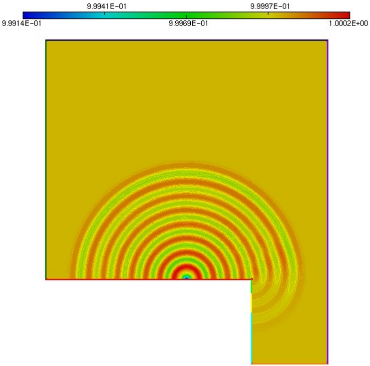

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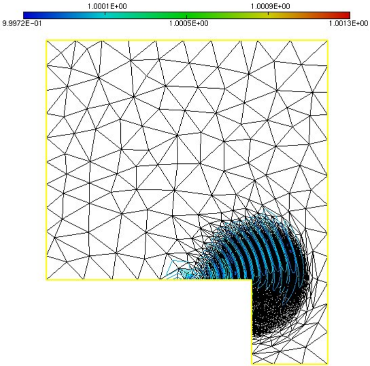

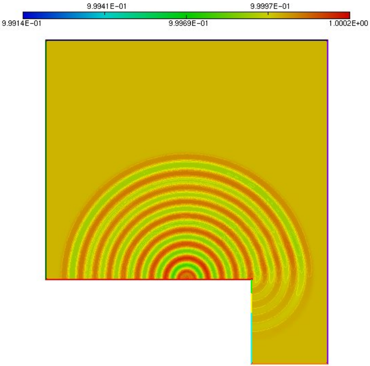

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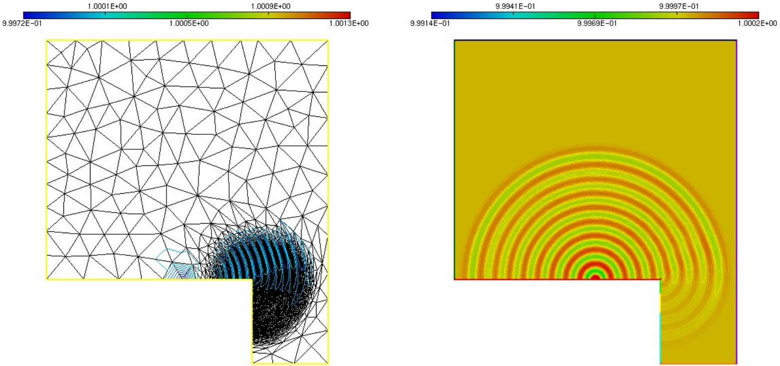

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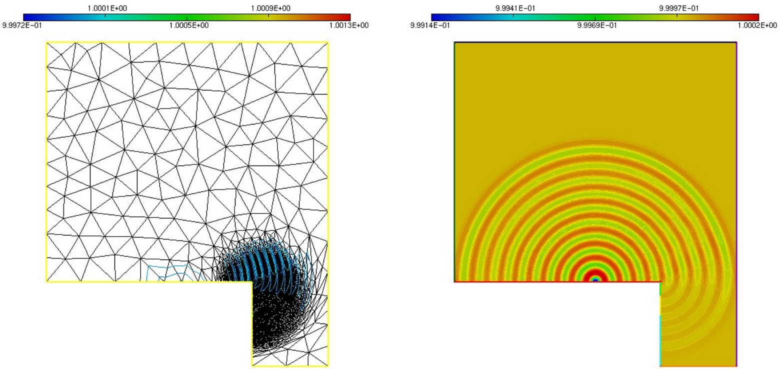

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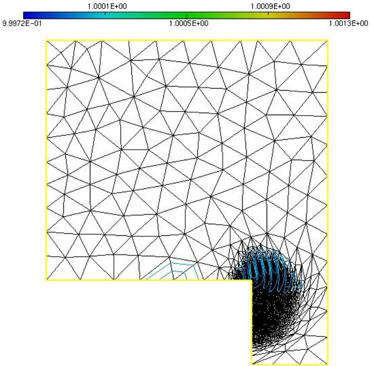

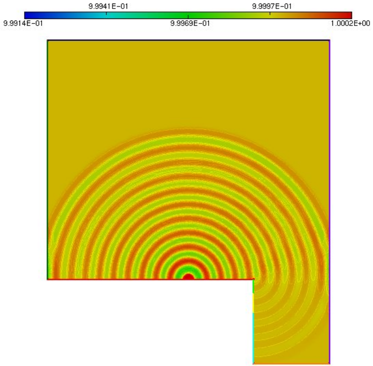

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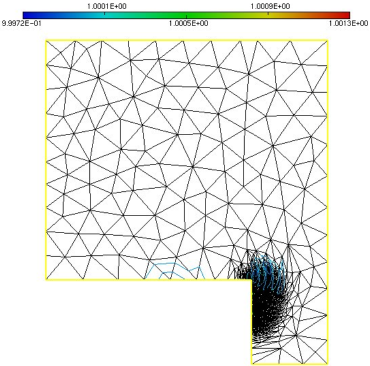

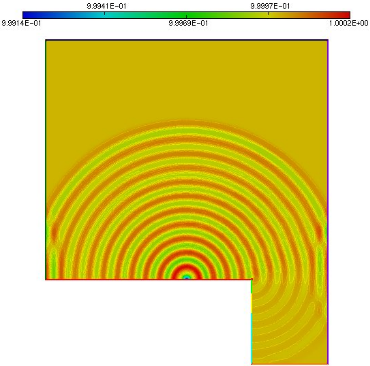

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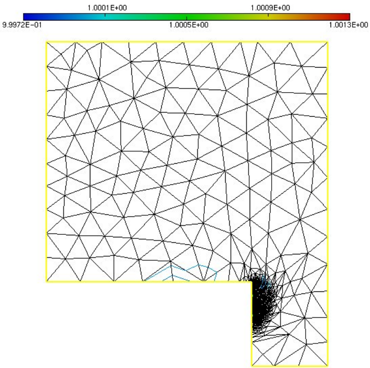

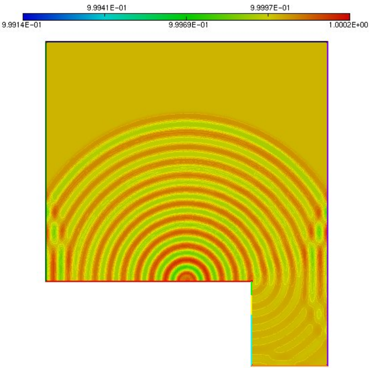

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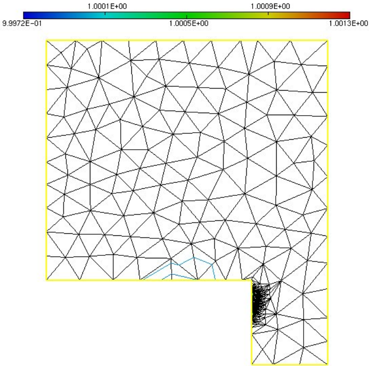
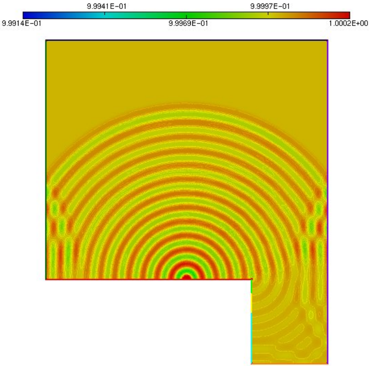
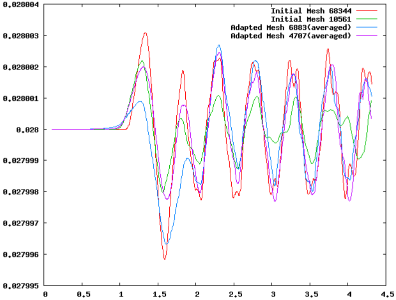
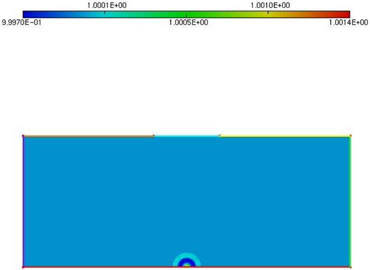




Application to acoustics

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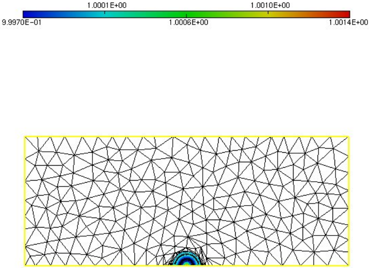
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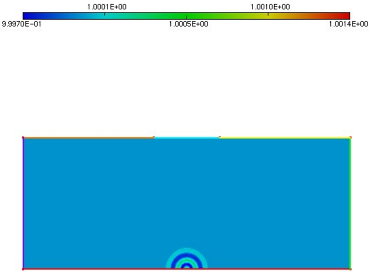
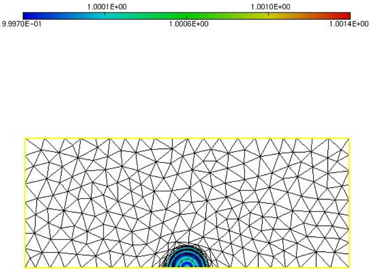
Application to 2D acoustics(2)





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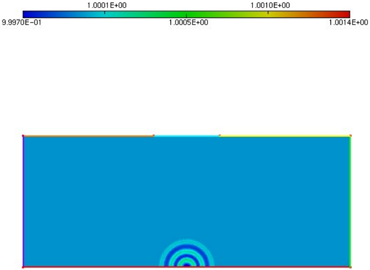

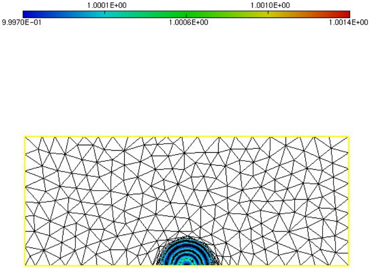

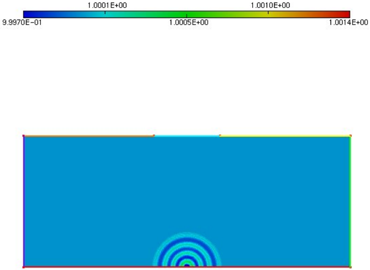
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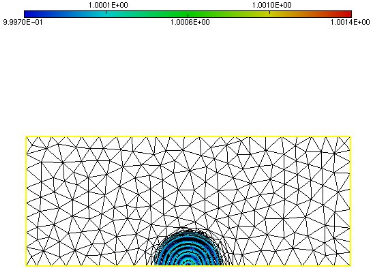
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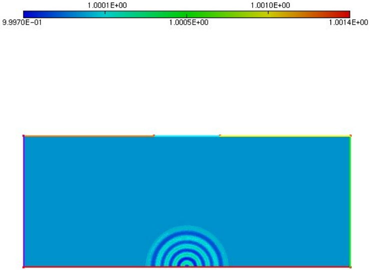

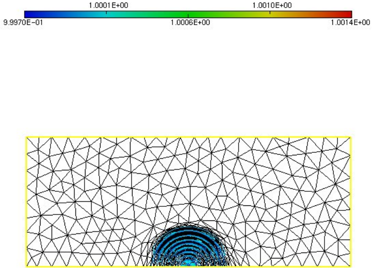

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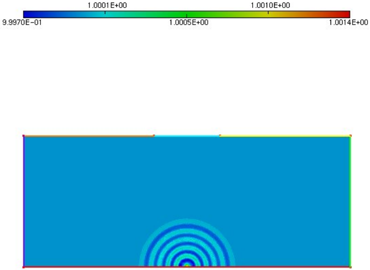

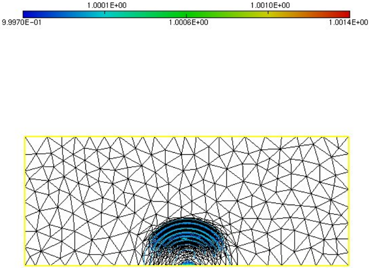

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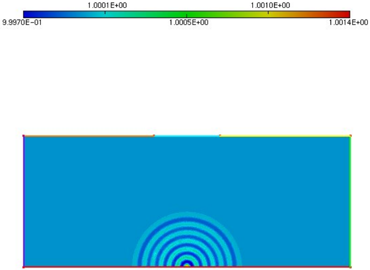

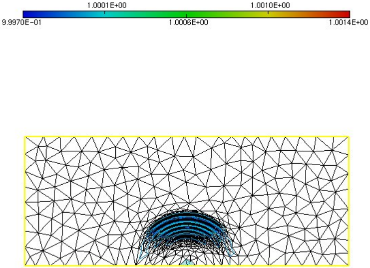

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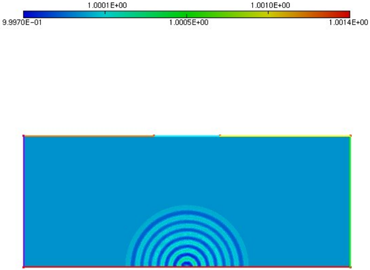

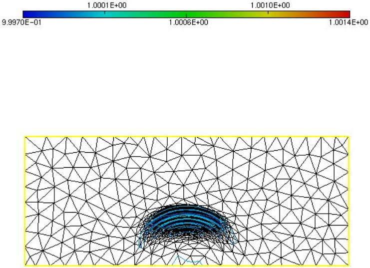

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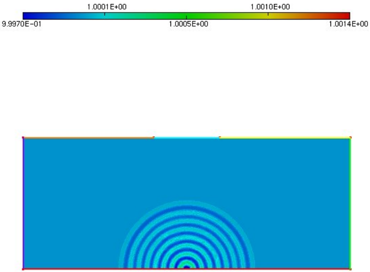

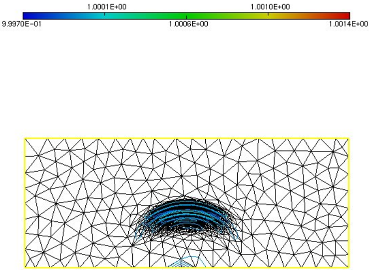

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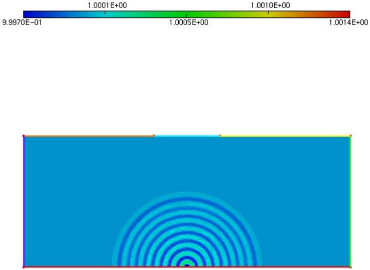

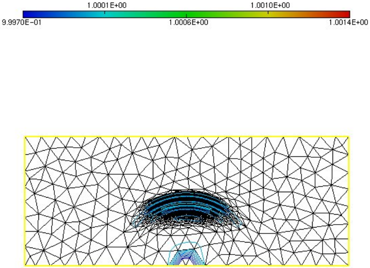

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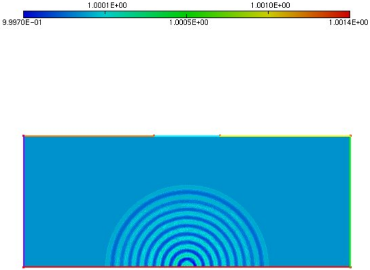

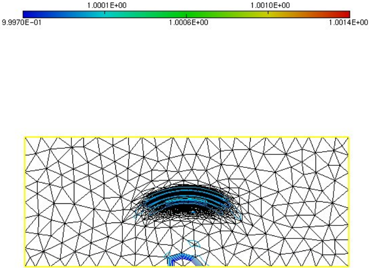

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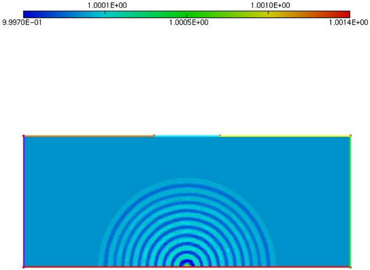

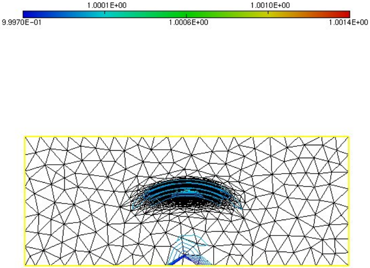

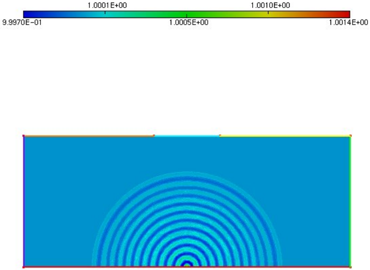
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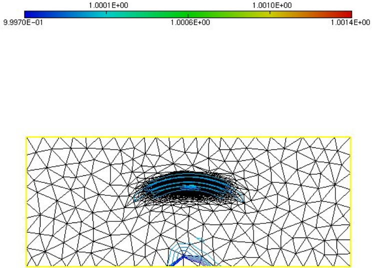
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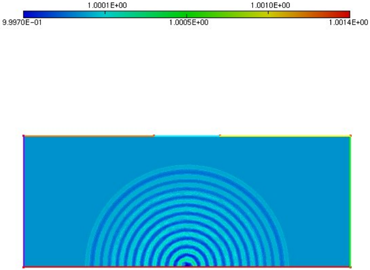

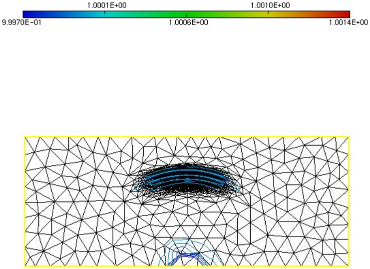

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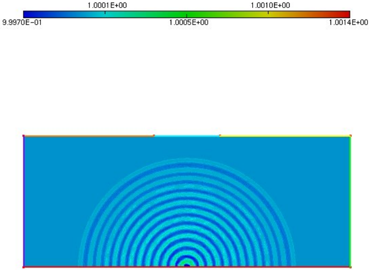

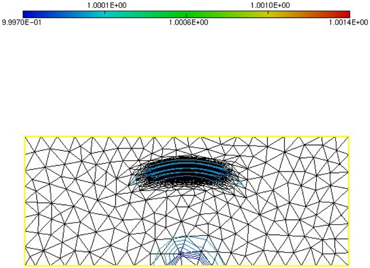

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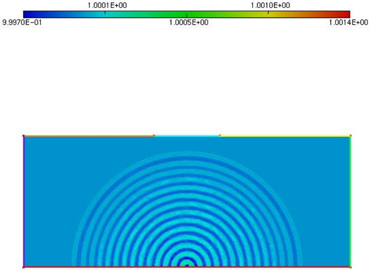

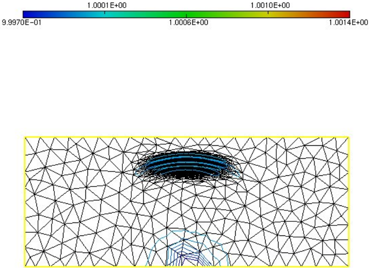

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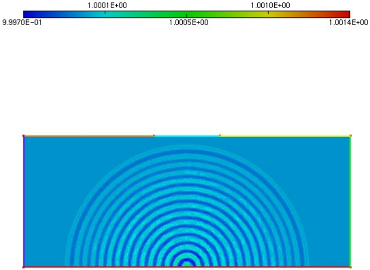

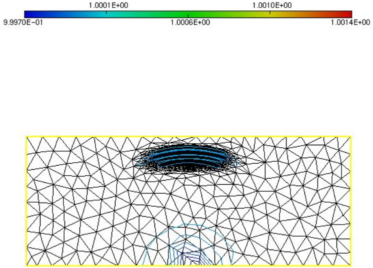

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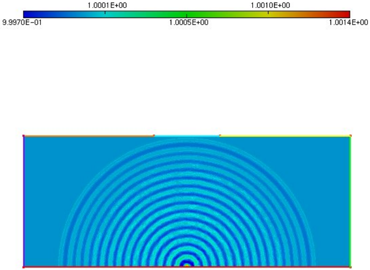

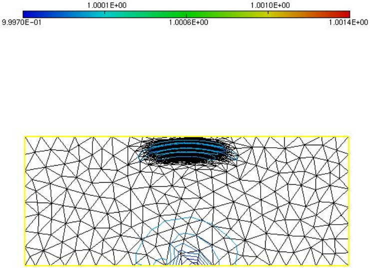

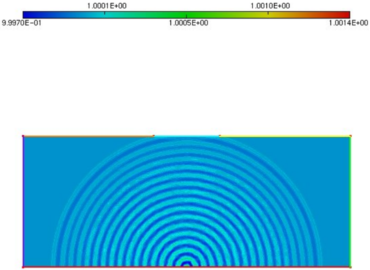
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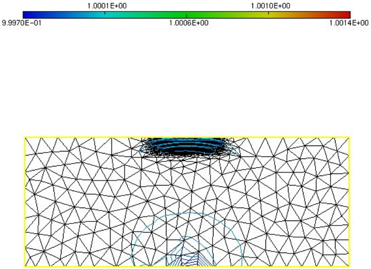
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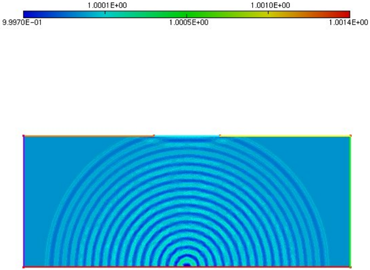
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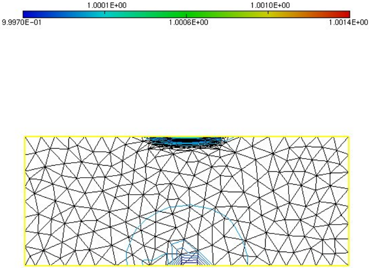
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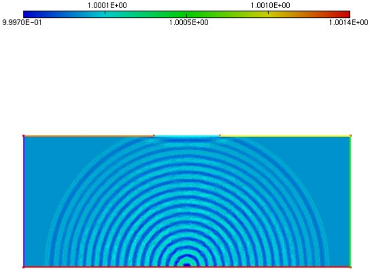
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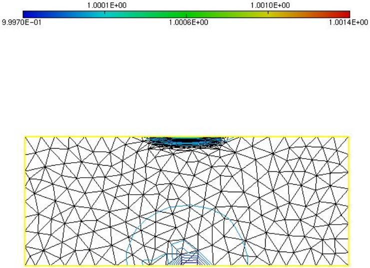
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Conclusion:

- New mesh adaptation algorithm which prescribes the spatial mesh of an unsteady simulation as the optimum of a goal-oriented error analysis;
- Extension to unsteadiness is applied in an implicit mesh-solution coupling which needs a non-linear iteration, the fixed point;
- The new algorithm is applied to a blast wave test case and a noise propagation test case and shows on these calculations the favourable behavior expected from an adjoint-based method (automatic selection of the mesh necessary for the target output)

Perpectives:

- Accurate integration of time errors in the mesh adaptation process with a more general formulation of the mesh optimisation problem (work in progress)
- Higher order adjoint schemes and 3D unsteady test-cases (work in progress)
- Application to turbulent aeroacoustics (3D Navier-Stokes equations)